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Introduction to the School

Setting the stage

1. The Stellar Atmosphere Physical System

A star as a ***gravitationally bounded***

open concentration *of*

matter *and* ***energy***

According to ***thermodynamics*** *a system may be:*

- ***open: matter and energy fluxes***
- ***close: only energy flux***
- ***isolated: absence of fluxes***

Fundamental fact: ***stars emit radiant energy***

Evidence of ***mass loss*** *from the observations of*

solar and stellar winds

Hence, definition of

Stellar atmosphere:

permeable boundary *through which*

both ***photons*** *and* ***particles*** *can* ***escape outwards***

Two components:

- ***radiation field***
- ***matter***

For both either a ***macroscopic*** *or a* ***microscopic*** *picture*

fluxes → ***gradients***

→ ***departure from equilibrium***

→ **TRANSPORT PROCESSES**

Radiative Transfer

The protagonist of this School

Stellar winds are the signature of

departure from hydrostatic equilibrium

hence ***transport of matter***

Supersonic velocity fields in early-type stars have important effects on RT in spectral line via Doppler effect

Line shifts and P Cyg profiles

- ***influence the mechanical structure of the outer layers;***
- ***provide useful diagnostics tools.***

Convective transport ***in*** ***late-type stars***
plays a fundamental role in

- ***the energy balance***
- ***the generation and advection of magnetic field***

The ***structure*** *of any physical system is shaped by the the* ***mutal interactions*** *among its components*

The structure is described by the values at any point of

- ***macroscopic*** *(thermodynamic)* ***variables*** *like T, P, ρ*
- *the* ***velocity*** *of the (ideal)* ***matter elements***

$$\text{physical relations among} \left\{ \begin{array}{c} \textbf{variables} \\ \textbf{constraints} \\ \textbf{internal energy} \end{array} \right\}$$

$$\text{expressed by} \left\{ \begin{array}{c} \textbf{conservation} \\ \textbf{state} \\ \textbf{transport} \end{array} \right\} \text{equations}$$

Tenet:

The physics of the problem dictates

the most effective algoritm for its solution

The ***algorithm*** *is a* ***numerical representation***

of the ***physical process***


A visit to the province of
Radiative Transfer

The dual nature of light

- ***Huygens: wave hypothesis (1690)***
- ***Newton: corpuscular hypothesis (Opticks, 1704)***
- ***Experiments by Young (1801) and Fresnel (1817)***
- ***Maxwell: electromagnetic waves (1873), revealed by Hertz (1888)***
- ***Einstein: quantum of light (1905)***

$$I(\mathbf{r},t;\mathbf{n},\nu) = h\nu\, c\, f(\mathbf{r},t;\mathbf{n},\nu)$$

macroscopic representation ⟷ ***corpuscular picture***

↓

$$I_0 = \frac{c}{8\pi} E_0^2$$

← ***wave picture***

wave – particle duality

2. Fluid dynamic - like picture

Picture based on macroscopic quantities

related to the microscopic photon picture

(corpuscular model of the radiation field)

Analogue with fluid dynamics:

flux of photons propagating along the

***paths of geometrical optics* (eikonal equation)**

that carry on and exchange energy with matter particles

Ray :

*amount of **radiant energy** of frequency* ν

*carried on along the **direction*** $\mathbf{n}$ *with speed* c

*per **unit time***

*across a **unit surface** perpendicular to* $\mathbf{n}$

rays $\Longleftrightarrow$ ***transport of energy***

Under the assumptions of

a weak electromagnetic field and
propagation through a diluted medium

the energy carried on by rays obeys the empirical laws of

radiometry

1. ***propagation through vacuum along straight lines with speed c***
2. ***all rays through a given point are independent***
3. ***they are linearly additive both in direction and frequency***

Hypotheses already formulated by Newton in his Opticks (1704)

The above laws of photometry warrants that

the ***transport process*** *is* ***intrinsically linear***

However

a ***single directed*** *quantity (i.e. a vector)* ***is not enough***
to specify ***completely*** *the radiation firld:*

virtually infinite pencil of rays

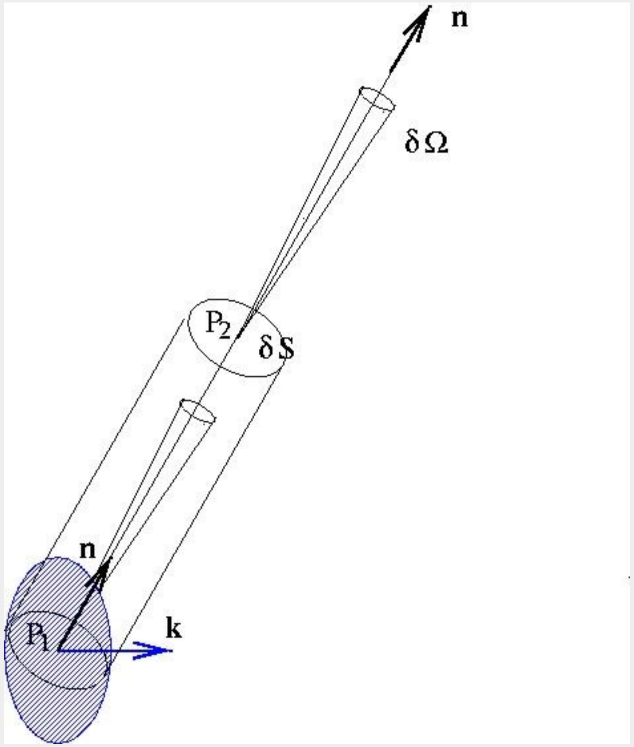
From rays to specific intensity

Fundamental physical observable in radiative tyranfer:

the energy carried on by a ray

→ ***Scalar*** *macroscopic* ***local*** *and* ***directed*** *quantity :*

specific intensity of the radiation field



observable:

amount of energy $\delta E_\nu(\boldsymbol{n})$

elements of the measure:

oriented surface $\boldsymbol{k}\,\delta S$ *around* P_1

solid angle $\delta\Omega$ *around* $\boldsymbol{n}$

time interval δt *spectral range* $\delta\nu$

$$\delta E_\nu(\boldsymbol{n}) \propto \boldsymbol{n}\cdot\boldsymbol{k}\,\delta S \;\; \delta\Omega \;\; \delta\nu \;\; \delta t$$

$$(\boldsymbol{n}\cdot\boldsymbol{k})^{-1} \lim \delta S\ \delta\Omega\ \delta\nu\ \delta t \rightarrow 0 \;\; \frac{\delta E_\nu(\boldsymbol{n})}{\delta S\ \delta\Omega\ \delta\nu\ \delta t} \equiv I(\boldsymbol{r}, t; \boldsymbol{n}, \nu)$$

By definition the specific intensity $I\left(\mathbf{r}, t; \mathbf{n}, \nu\right)$

characteyerized by $\left(\mathbf{n}, \nu\right)$

is the coefficient of proportionality

between the

observable and the elements of the measurement

Dimension:

$$[I] = \left(M L^2 T^{-2}\right) \cdot L^{-2} \cdot T^{-1} \cdot T = M T^{-2}$$

i.e. ***energy flux per unit time and unit frequency band***

3. The RT equation as a kinetic equation for photons

Photon distribution function ***(microscopic picture)***

Because of the ***corpuscular nature*** *of photons, let us define a*

distribution function ***such that***

$$f\left(\mathbf{r},t;\mathbf{n},\nu\right)\,d\Omega\,d\nu$$

is equal to the

nr. of photons per unit volume *at* **r** *and t*

in the band $\left(\nu,\nu+d\nu\right)$

that propagates along **n** *with speed c into* $d\Omega$.

f is characterized by the pair $\left(\mathbf{n};\nu\right)$

directed *and* ***spectral*** *quantity*

$$[f] = L^{-3}\cdot T$$

Transport process *in terms of the* ***photon distribution function***

***Specific energy** flowing through* $\boldsymbol{k}\cdot\boldsymbol{n}\,dS$

$$d\,E_\nu(\boldsymbol{n}) \;=\; h\nu\; f(\boldsymbol{r},t;\boldsymbol{n},\nu)\;\boldsymbol{n}\cdot\boldsymbol{k}\;dS\;c\,dt\;\,d\Omega\;d\nu$$

By direct comparison

$$I(\boldsymbol{r},t;\boldsymbol{n},\nu) \;=\; c\,h\nu\; f(\boldsymbol{r},t;\boldsymbol{n},\nu)$$

A ***kinetic equation*** *for any* ***transported quantity*** *is formally*

Total rate of change = Source terms – Sink terms

(Boltzmann's equation)

Total rate of change = Eulerian derivative

In our case

Sources and sinks determined by:

atomic properties of the interaction matter - radiation

equation of state of matter (LTE) or SE equations

Distribution function $F(\mathbf{r},\mathbf{p},t)$ ***for photons with momentum***

$$\mathbf{p} = \mathbf{n}\frac{h\nu}{c} \quad ; \qquad \mathbf{p} = \mathbf{p}(\mathbf{n},\nu)$$

Kinetic equation:

$$\frac{d}{dt}F(\mathbf{r},\mathbf{p},t) = \left[\frac{\delta F}{\delta t}\right]_{sources} - \left[\frac{\delta F}{\delta t}\right]_{sinks}$$

It can be shown that

$$f(\mathbf{r},t;\mathbf{n},\nu) = \frac{h^3\nu^2}{c^3}F(\mathbf{r},\mathbf{p},t)$$

Previusly:

$$I(\mathbf{r},t;\mathbf{n},\nu) = ch\nu\, f(\mathbf{r},t;\mathbf{n},\nu)$$

parameters $(\mathbf{n},\nu)$

$$\rightarrow \qquad F(\mathbf{r},\mathbf{p},t) = \frac{c^2}{h^4\nu^3}I(\mathbf{r},t:\mathbf{n},\nu)$$

Total derivative:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}} + \dot{\boldsymbol{p}}\cdot\frac{\partial}{\partial \boldsymbol{p}} \;; \qquad \frac{\partial}{\partial \boldsymbol{r}} = \nabla$$

For photons $\boldsymbol{v} = c\,\boldsymbol{n}$ *and* $\dot{\boldsymbol{p}} = 0$

time – space evolution

$$\rightarrow \quad \frac{1}{c}\frac{\partial}{\partial t} I(\boldsymbol{r},t:\boldsymbol{n},\nu) + \boldsymbol{n}\cdot\nabla I(\boldsymbol{r},t;\boldsymbol{n},\nu) =$$

$$\frac{1}{c}\left[\frac{\delta I}{\delta t}\right]_{sources} - \frac{1}{c}\left[\frac{\delta I}{\delta t}\right]_{sinks} = \left[\frac{\delta I}{\delta l}\right]_{sources} - \left[\frac{\delta I}{\delta l}\right]_{sinks}$$

physical properties

where $\delta l = c\,\delta t$ *is a* ***path length*** *along* $\boldsymbol{n}$

The Radiative Transfer equation

$$\frac{1}{c}\,\frac{\partial}{\partial t}\,I\left(\boldsymbol{r},t:\boldsymbol{n},\nu\right)+\boldsymbol{n}\cdot\nabla\,I\left(\boldsymbol{r},t\,;\boldsymbol{n},\nu\right)\;=\;\left[\frac{\delta I}{\delta l}\right]_{sources}-\left[\frac{\delta I}{\delta l}\right]_{sinks}$$

A. Schuster (1851 - 1934)

K. Schwarzschild (1873 - 1916)

mathematical formulation *of a* ***directional problem***

in terms of the ***macroscopic quantity*** ***specific intensity***

For ***any*** ***specific intensity*** *,*

characterized *by the* ***pair of parameters*** $(\mathbf{n}\,;\nu)$,

one specific RT equation

Each term in the RT equation has dimension

$$\left(M\,L^{2}T^{2}\right)L^{-2}\,L^{-1} \;=\; M\,L^{-1}\,T^{-2}$$

energy flux per unit length

5. Macroscopic RT coefficients and the source function

Consistently with the macroscopic picture

we consider ***homogeneous volume elements*** *that* ***emit*** *and* ***absorb*** *radiant energy* ***isotropically***

All the physical information *at* ***atomic level*** *is incorporated into a* ***limited number*** *of* ***macroscopic coefficients***

Thermal emission coefficient

ΔE_ν^{th} *energy emitted along* **n** *by* ΔV

into $\Delta \Omega$

during Δt

in $(\nu, \nu + \Delta \nu)$

measurable quantity

parameters of the measure

$$\Delta E_\nu^{th} \propto \Delta V \; \Delta \Omega \; \Delta \nu \; \Delta t$$

$$\lim_{\Delta\sigma \; \Delta\Omega \; \Delta\nu \; \Delta t \to 0} \frac{\delta E_\nu^{th}}{\delta V \; \delta \Omega \; \delta \nu \; \delta t} \equiv \eta_\nu^{th}$$

$$\left[\eta_\nu^{th}\right] = M L^{-1} T^{-2}$$

Decrease of the specific intensity

along a path δl *in the direction* **n**

True absorption coefficient $a_\nu(\boldsymbol{n})$:

fraction of energy ***removed***

converted into internal energy

$$\delta I(\boldsymbol{n}) \propto I(\boldsymbol{n})\,\delta l \quad \rightarrow \quad \frac{\delta I(\boldsymbol{n})}{I(\boldsymbol{n})} = a_\nu(\boldsymbol{n})\,\delta l$$

Likewise

Scattering coefficient $\sigma_\nu(\boldsymbol{n})$:

fraction of energy ***removed***

diverted into a different direction

Extinction coefficient

Global *effect of the* ***attenuation****,*
i.e., removal of photons from a given direction ***n***

$$\chi_\nu(\boldsymbol{n}) \equiv a_\nu(\boldsymbol{n}) + \sigma_\nu(\boldsymbol{n})$$

$$[\chi_\nu] = [a_\nu] = [\sigma_\nu] = L^{-1}$$

Total emission coefficient

diverted from ***n****' into* ***n***

$$\eta_\nu(\boldsymbol{n}) = \eta_\nu^{th} + \oint \sigma_\nu(\boldsymbol{n}')\, p(\boldsymbol{n}', \boldsymbol{n})\, d\boldsymbol{n}'$$

thermal contribution

The source function

$$S_\nu(\boldsymbol{n}) \equiv \frac{\eta_\nu(\boldsymbol{n})}{\chi_\nu(\boldsymbol{n})}$$

$$[S_\nu] = \left(M L^{-1} T^{-2}\right) / L^{-1} = M T^{-2} = [I_\nu(\boldsymbol{n})] = [B_\nu]$$

TE (or LTE) → ***isotropic scattering***

$$\eta_\nu = a_\nu B_\nu(T) \qquad \eta_\nu^s = \sigma_\nu \frac{1}{4\pi} \oint I(\boldsymbol{n}') d\boldsymbol{n}' = \sigma_\nu J_\nu$$

$$S_\nu = \frac{a_\nu B_\nu(T)}{a_\nu + \sigma_\nu} + \frac{\sigma_\nu J_\nu}{a_\nu + \sigma_\nu} = \varepsilon_\nu B_\nu(T) + (1 - \varepsilon_\nu) J_\nu$$

$$\varepsilon_\nu \equiv \frac{a_\nu}{a_\nu + \sigma_\nu}$$

If the ***sources and sinks*** *are* ***given****.*

the RT equation

$$\frac{1}{c}\frac{\partial}{\partial t} I(\boldsymbol{r},t:\boldsymbol{n},\nu) + \boldsymbol{n}\cdot\nabla I(\boldsymbol{r},t;\boldsymbol{n},\nu) = \left[\frac{\delta I}{\delta l}\right]_{sources} - \left[\frac{\delta I}{\delta l}\right]_{sinks}$$

is a

1st order linear ODE

From a linear to non-linear problem

Mathematical complications *arise when*
the individual specific RT equations are coupled
through the source function

Non-local problems

Moreover transport processes necessarily implies
non-local effects

Illustrative examples:

$$a_\nu = 0$$

pure scattering

$$S_\nu = J_\nu = \frac{1}{4\pi} \oint I_\nu(\boldsymbol{n})\, d\boldsymbol{n}$$

internal mechanism

$$\sigma_\nu = 0$$

pure absorption

$$S_\nu = a_\nu B_\nu(T) = S_\nu(T)$$

energy balance: $$\int_0^\infty a_\nu B_\nu(T)\, d\nu = \int_0^\infty a_\nu J_\nu\, d\nu$$

$$\rightarrow T$$

external mechanism *given by a* ***conservation equation***

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Back to electrodynamics homeland

1. Electrodynamic formulation

James Clark Maxwell
(1831 - 1879)

$$\nabla \cdot \boldsymbol{D} = 4\pi\rho$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = 0$$

$$\nabla \times \boldsymbol{H} + \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} = \frac{4\pi}{c}\boldsymbol{J}$$

$$\boldsymbol{J} = \rho\, \boldsymbol{v}$$

A Dynamical Theory of the Electromagnetic Field (1865)

Gauss conventional system of units:

$$[E] = [D] = [B] = [H] = M^{1/2}L^{-1/2}T^{-1}$$

$$[J] = M^{1/2}L^{-1/2}T^{-2}$$

$\varepsilon = \mu = 1$ ***dimensionless***

Following Maxwell:

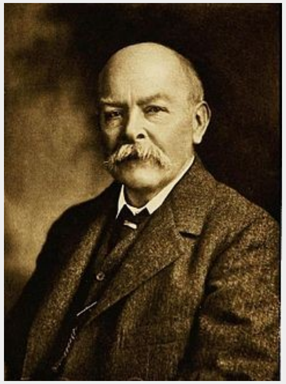
magnetic and electric energy density

$$W_{mag} = \frac{1}{8\pi}\,\mathbf{H}\cdot\mathbf{B} \;\; ; \qquad W_{elec} = \frac{1}{8\pi}\,\mathbf{D}\cdot\mathbf{E}$$

$$W \equiv W_{mag} + W_{elec}$$

energy is localized in the field

John H. Poynting
(1852 – 19 14)



Poynting's vector: $$\boldsymbol{S} \equiv \frac{c}{4\pi} \boldsymbol{E} \times \boldsymbol{H}$$

$$[S] = \left(L\,T^{-1}\right)\left(M^{1/2}L^{-1/2}T^{-1}\right)^2 = \left(M\,L^2\,T^{-2}\right)T^{-1}\,L^{-2}$$

i.e. $$\frac{energy}{time \cdot surface} = power\ flux$$

By a proper treatment of the last two Maxwell's equations

$$\rightarrow \quad \frac{1}{4\pi}\boldsymbol{H}\cdot\dot{\boldsymbol{B}} + \frac{1}{4\pi}\boldsymbol{E}\cdot\dot{\boldsymbol{D}} + \boldsymbol{E}\cdot\boldsymbol{J} + \nabla\boldsymbol{S} = 0$$

Poynting's theorem

$$[each\ term] = M\,L^{-1}T^{-3} = \left(M\,L^2T^{-2}\right)T^{-1}\,L^{-3}$$ i.e. **power density**

Physical meaning of the Poynting's vector:

energy flux per unit time
across ***unit area*** *of the*
boundary surface of the volume considered

→ ***Transport of energy of the electromagnetic field***

The Poynting's ***vector*** *accounts for the*
intrinsic directed aspect
of the ***propagation of the electromagnetic field***.

It can be shown that:

$$\dot{W}_{elec} = \frac{1}{8\pi}\boldsymbol{E}\cdot\dot{\boldsymbol{D}} + \frac{1}{8\pi}\dot{\boldsymbol{E}}\cdot\boldsymbol{D} = \frac{1}{4\pi}\boldsymbol{E}\,\dot{\boldsymbol{D}}$$

The same for $\dot{W}_{mag}$

$$W_J \equiv \boldsymbol{E}\cdot\boldsymbol{J} \quad \textit{Joule heat}$$

From the Poynting's theorem:

$$\dot{W} + \nabla\cdot\boldsymbol{S} = -W_J$$

energy balance of the electromagnetic field

By integration over V and Gauss theorem:

$$\int_{\Sigma}\boldsymbol{S}\cdot\boldsymbol{n}\,d\sigma = -\int_{V}\left[\frac{\partial W}{\partial t} + W_J\right]dV$$

conservation equation

Transport of radiant energy by an e.m. wave

monochromatic polarized plane wave
propagating along the x – axis, specified by $\hat{\boldsymbol{x}}$

$\boldsymbol{E} \perp \boldsymbol{H}$: *only* E_y *and* H_z *not* 0

$$\frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2} - \frac{\partial^2 E_y}{\partial x^2} = 0$$ *wave equation*

solution $E_y(x,t) = E_0 \cos(k x - \omega t)$

by proper manipulation

$$\frac{\partial}{\partial t}\left\{ \frac{1}{8\pi k^2}\left[\frac{1}{c^2}\left(\frac{\partial E_y}{\partial t}\right)^2 + \left(\frac{\partial E_y}{\partial x}\right)^2 \right]\right\} - \frac{\partial}{\partial x}\left(\frac{1}{4\pi k^2}\frac{\partial E_y}{\partial t}\frac{\partial E_y}{\partial x}\right) = 0 \ .$$

$$e \equiv \frac{1}{8\pi k^2}\left[\frac{1}{c^2}\left(\frac{\partial E_y}{\partial t}\right)^2 + \left(\frac{\partial E_y}{\partial x}\right)^2 \right] \qquad f \equiv -\left(\frac{1}{4\pi k^2}\frac{\partial E_y}{\partial t}\frac{\partial E_y}{\partial x}\right)$$

From the previuos definitions:

$$\frac{\partial e}{\partial t} + \frac{\partial f}{\partial x} = 0 \ .$$

wave equation => ***equation of continuity***

$$[e] = \left(M L^2 T^{-2}\right) L^{-3} \ ; \qquad [f] = \left(M L^2 T^{-2}\right) T^{-1} L^{-2}$$

energy density ***power flux***

$$e(t) = \frac{E_0^{\ 2}}{4\pi} \sin^2(kx - \omega t)$$

$$W(t) = W_{elec}(t) + W_{mag}(t) = \frac{E_0^{\ 2}}{4\pi} \cos^2(kx - \omega t)$$

$$e \iff W$$

$$f(t) = \frac{E_0^{\ 2}}{4\pi} \frac{\omega}{k} \sin^2(kx - \omega t) = \frac{c}{4\pi} E_0^{\ 2} \sin^2(kx - \omega t)$$

$$\boldsymbol{S}(t) = \frac{c}{4\pi} E_y^2(t)\, \hat{\boldsymbol{x}} = \frac{c}{4\pi} E_0^{\ 2} \cos^2(kx - \omega t)\, \hat{\boldsymbol{x}} \ .$$

$$f \ \hat{\boldsymbol{x}} \iff \boldsymbol{S}$$

7. Electrodynamical vs. macroscopical picture

Energy density of the radiation field

In the time interval dt ***the volume*** $dV = \mathbf{n}\cdot\mathbf{k}\; dS\; c\; dt$ ***is filled in by radiant energy***

Specific energy density: $U(\mathbf{r}, t; \mathbf{n}, \nu) \equiv \dfrac{d E_\nu(\mathbf{n})}{dV}$

directed *and* ***spectral***

By definition $U(\mathbf{r}, t; \mathbf{n}, \nu)\, d\Omega\, d\nu = \dfrac{1}{c} I(\mathbf{r}, t; \mathbf{n}, \nu)\, d\Omega\, d\nu$

By integration over all the directions

$$u_\nu \equiv u(\mathbf{r}, t; \nu) \equiv \frac{1}{c} \oint I(\mathbf{r}, t; \mathbf{n}, \nu) d\Omega = \frac{4\pi}{c} J(\mathbf{r}, t; \nu)$$

→ ***spectral***

$$[u_\nu] = (M L^2 T^{-2}) L^{-3} T$$

Correspondence *between*

the ***specific intensity*** *and the* ***electric field strength***

Monochromatic plane wave *of frequency* $\nu_0 = 1/T$

propagating along $\boldsymbol{n}_0 = \boldsymbol{n}_0(\theta_0, \phi_0)$

$$\boldsymbol{n}_0 \equiv \hat{x} \; ; \qquad \hat{x} \perp \boldsymbol{E} \perp \boldsymbol{H}$$

$$[\boldsymbol{E}] = [\boldsymbol{D}] = [\boldsymbol{B}] = [\boldsymbol{H}]$$

The solution of the wave equation

$$\frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2} - \frac{\partial^2 E_y}{\partial x^2} = 0$$

is $$E_y(x,t) = E_0 \cos(kx - \omega t)$$

From the average over T of $$W_{elec} \equiv \frac{1}{8\pi}\boldsymbol{E}\cdot\boldsymbol{D}$$

and $$W_{mag} \equiv \frac{1}{8\pi}\boldsymbol{H}\cdot\boldsymbol{B} \ .$$

=> $$\langle W(t)\rangle_T = \frac{E_0^2}{8\pi} \ .$$

Corresponding specific intensity :

$$I\left(\vartheta,\phi,\nu\right)=I_0\,\delta\left(\vartheta-\vartheta_0\right)\delta\left(\phi-\phi_0\right)\delta\left(\nu-\nu_0\right)$$

$$[I]\;=\;M\,T^{-2}\;;\quad[\delta(\nu-\nu_0)]\;=\;T\;;\quad[I_0]\;=\;M\,T^{-3}$$

From the physical standpoint

$$\left\langle W(t)\right\rangle_T\;=\;\frac{E_0^2}{8\pi}\;=\;u(\boldsymbol{r},t)$$

$$u(\boldsymbol{r},t)\;=\;\frac{I_0}{c}\int_0^\infty d\nu\;\delta(\nu-\nu_0)\oint d\Omega\;\delta(\theta-\theta_0)\,\delta(\phi-\phi_0)\;=\;\frac{I_0}{c}$$

$$\Rightarrow\qquad I_0\;=\;\frac{c}{8\,\pi}\,E_0^2$$

$$[I_0]\;=\;[c\,E_0^2]\;=\;M\,T^{-3}$$ ***power flux***

Moments of the specific intensity

0 th order moment: ***average mean intensity***

$$J(\boldsymbol{r},t;\nu) \equiv \frac{1}{4\pi}\oint I(\boldsymbol{r},t;\boldsymbol{n},\nu)\, d\boldsymbol{n} = \frac{c}{4\pi} u_\nu$$

scalar

1 st order moment: ***flux of radiation***

$$\boldsymbol{F}_\nu(\boldsymbol{r},t) \equiv \oint I(\boldsymbol{r},t;\boldsymbol{n},\nu)\, \boldsymbol{n}\, d\boldsymbol{n}$$

vector

2 nd order moment: ***radiation pressure***

$$\underline{\underline{\boldsymbol{T}_\nu}}(\boldsymbol{r},t) \equiv \frac{1}{c}\oint I(\boldsymbol{r},t;\boldsymbol{n},\nu)\, \boldsymbol{n}\boldsymbol{n}\, d\boldsymbol{n}$$

tensor

(dyadic notation)

Electromagnetic counter part of $\mathbf{F}_\nu(\mathbf{r},t)$

$$\mathbf{F}_\nu(\mathbf{r},t) \equiv \oint I(\mathbf{r},t;\mathbf{n},\nu)\,\mathbf{n}\,d\mathbf{n}$$

is the ***monochromatic power flux*** *of the radiation field*

$$\left(M\,L^2 T^{-2}\right) T^{-1}\,L^{-2}\,T = M\,T^{-2}$$

$$E_y(x,t) = E_0\cos(kx - \omega t)\,; \qquad \hat{x}\perp\mathbf{E}\perp\mathbf{H}\,; \qquad |E_0| = |H_0|$$

$$\int_0^\infty d\nu \oint d\mathbf{n}\,I(\mathbf{r},t;\mathbf{n},\nu)\,\mathbf{n} = I_0\,\mathbf{n_0} = \frac{c}{8\pi}E_0^2\,\mathbf{n_0}$$

bolometric vector flux

power flux

Poynting vector: $\langle \mathbf{S}(t)\rangle_T = \dfrac{c}{8\pi}E_0^2\,\mathbf{n_0}$

(time averaged)

Correspondence of the ***radiative pressure*** *with the* ***Maxwell stress tensor***

$$\boldsymbol{p}(\boldsymbol{n},\nu) = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \frac{h\nu}{c}\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$$

moment *carried on by a photon* $(\boldsymbol{n},\nu)$

radiative pressure tensor:

$$\underline{\underline{\boldsymbol{T}_{\boldsymbol{\nu}}}}(\boldsymbol{r},t) \equiv \frac{1}{c}\oint I(\boldsymbol{r},t;\boldsymbol{n},\nu)\,\boldsymbol{n}\boldsymbol{n}\,d\boldsymbol{n}$$

$$\left[\underline{\underline{\boldsymbol{T}_{\boldsymbol{\nu}}}}\right] = \left(M\,L\,T^{-1}\right)L^{-2}$$

flux of momentum

net transport *of* $\boldsymbol{p}$:

$$\boldsymbol{n} \cdot \frac{1}{c} \boldsymbol{F}_\nu = \boldsymbol{n} \cdot \oint \frac{h\nu}{c} f(\boldsymbol{r}, t; \boldsymbol{n}, \nu)\, c\boldsymbol{n}\, d\boldsymbol{n}$$

$$\boldsymbol{G}_\nu(\boldsymbol{r}, t) \equiv \frac{1}{c^2} \boldsymbol{F}_\nu(\boldsymbol{r}, t)$$

monochromatic momentum density

of the radiation field

$$[G_\nu] = (M\, T^{-2})\, L^{-2}\, T^2 = (M\, L\, T^{-1})\, L^{-3}\, T$$

$$\boldsymbol{G} \equiv \int_0^\infty \boldsymbol{G}_\nu d\nu = \frac{1}{c^2} \int_0^\infty \boldsymbol{F}_\nu\, d\nu = \frac{1}{c^2} \boldsymbol{S}$$

$\boldsymbol{S}$ *bolometric vector flux*

For each component:

$$\frac{\partial(\mathbf{G}_\nu)_j}{\partial t} = \frac{1}{c^2}\frac{\partial(\mathbf{F}_\nu)_j}{\partial t} = -\nabla\cdot\left[(\mathbf{G}_\nu)_j\, c\, \mathbf{n}\right]$$

continuity equation

From previous results:

$$\frac{\partial}{\partial t}\int_0^\infty \mathbf{G}_\nu\, d\nu = -\nabla\cdot\int_0^\infty \underline{\underline{\mathbf{T}}}_\nu\, d\nu \quad \Rightarrow \quad \frac{\partial \mathbf{G}}{\partial t} = -\nabla\cdot\underline{\underline{\mathbf{T}}}$$

$$\left(M\,L\,T^{-1}\right)L^{-3}\,T^{-1}$$

momentum density *of the* ***electromagnetic field****:*

$$\frac{\partial \mathbf{G}_{em}}{\partial t} \equiv \nabla\cdot\underline{\underline{\mathbf{T}}}^{M}$$

$$\left(M\,L\,T^{-1}\right)L^{-3}\,T^{-1}$$

$$\mathbf{G} \Longleftrightarrow \mathbf{G}_{em} \qquad \underline{\underline{\mathbf{T}}} = -\underline{\underline{\mathbf{T}}}^{M}$$


"That's all Folks!"