XXIX IAC Winter School of Astrophysics

APPLICATIONS OF

RADIATIVE TRANSFER

TO STELLAR AND PLANETARY ATMOSPHERES

Fundamental physical aspects of radiative transfer:
III.- Structure equations
IV.-Line broadening

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Outline

III. Structure Equations

- Momentum equation (hydrostatic equilibrium)
- Energy equation
  - Radiative equlibrium
  - Convection

IV. Line broadening

- Introduction
- Microscopic line broadening
  - Natural, collisional, thermal, microturbulence
- Hjerting and Voigt functions
- Macroscopic line broadening
  - Macroturbulence
  - Rotation

We have considered to solve equations of radiative transfer and the statistical equilibrium equations on a given density and temperature structure

Two more equations are needed to give density and temperature (plus the state equation)

- The momentum equation (or the hydrostatic equilibrium)
- The energy equation (or the radiative equilibrium equation)

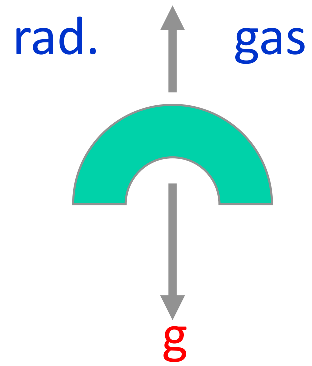
Momentum Equation

for a mass element in our atmosphere we will have

$$dm\frac{d\vec{\upsilon}}{dt} = \sum d\vec{f_i}$$

Using the momentum conservation (with the acceleration)

$$\frac{\partial\vec{\upsilon}}{\partial t} + \vec{\upsilon}\frac{\partial\vec{\upsilon}}{\partial r} = -\frac{1}{\rho}\frac{dP}{dr}\vec{r} - \frac{Gm_r}{r^2}\vec{r} + \overrightarrow{g_{rad}}$$



After some manipulation this can be written

(a being the sound velocity, assumed constant)

$$\underbrace{\left(1-\frac{a^2}{\upsilon^2}\right)\upsilon\frac{d\upsilon}{dr}}_{\text{velocity field}} = \underbrace{-\frac{da^2}{dr}+\frac{2a^2}{r}}_{\text{gas pressure}} \underbrace{-\frac{Gm_r}{r^2}}_{\text{gravity}} \underbrace{+g_{rad}}_{\text{radiation pressure}}$$

Momentum Equation

If the forces are in balance, $\upsilon = 0$, and we recover the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\rho(r)\left(g_{\mathrm{grav}} - g_{\mathrm{rad}}\right) = -\rho(r)\, g_{\mathrm{eff}}$$

Sometimes, the mass column is used

$$\frac{dP}{dm'} = -g_{\mathrm{eff}}$$

A simple idea is to consider a isotermic atmosphere with just gas pressure

$$dP_g = -g\rho dr \quad \Rightarrow \quad \frac{dP_g}{P_g} = -g\frac{\rho}{P_g}dr = -\frac{g\mu m_H}{k_B T}dr$$

integrating at constant T

$$\int_{P_0}^{P}\frac{dP_g}{P_g} = \int_{r_0}^{r_1} -\frac{g\mu m_H}{k_B T}dr \quad \Rightarrow \quad \frac{P}{P_0} = e^{-\frac{g\mu m_H}{k_B T}(r_1 - r_0)} = e^{-\Delta r/H} \quad \Rightarrow \quad P = P_0\, e^{-\Delta r/H}$$

with H the pressure scale height, the distance needed for the pressure to decrease $1/e$

$$H = \frac{k_B T}{g\mu m_H}$$

high g and low T make H small, i.e., the atmosphere more compact

A more compact atmosphere has more collisions and tends to higher opacities

Energy equation – Radiative equilibrium

Begin with the RTE

$$\mu \frac{dI_\nu}{dx} = -\chi_\nu \left(I_\nu - S_\nu \right)$$

integrating over angle and assuming that the source function does not depend on the direction:

$$\frac{d}{dx} \frac{1}{2} \int_{-1}^{+1} I_\nu \mu \, d\mu = -\chi_\nu \frac{1}{2} \int_{-1}^{+1} \left(I_\nu - S_\nu \right) d\mu \Rightarrow \frac{d}{dx} H_\nu = -\chi_\nu \left(J_\nu - S_\nu \right)$$

integrating now over frequency

$$\frac{d}{dx} \int_0^\infty H_\nu d\nu = 0 = -\int_0^\infty \chi_\nu \left(J_\nu - S_\nu \right) d\nu$$

Energy equation – Radiative equilibrium

The coupling with temperature is clear when we remember that we can write

$$S_\nu = \frac{\kappa_\nu}{\kappa_\nu + \sigma_\nu} B_\nu + \frac{\sigma_\nu}{\kappa_\nu + \sigma_\nu} J_\nu \quad (\text{with } \chi_\nu = \kappa_\nu + \sigma_\nu)$$

replacing S_ν we get

$$\int_0^\infty (\kappa_\nu + \sigma_\nu)\left(J_\nu - \frac{\kappa_\nu}{\kappa_\nu + \sigma_\nu} B_\nu - \frac{\sigma_\nu}{\kappa_\nu + \sigma_\nu} J_\nu \right) d\nu = 0 \Rightarrow \int_0^\infty (\kappa_\nu J_\nu + \sigma_\nu J_\nu - \kappa_\nu B_\nu - \sigma_\nu J_\nu)\, d\nu = 0$$

$$\int_0^\infty \kappa_\nu \left(J_\nu - B_\nu \right) d\nu = 0$$

(note: scattering plays no role in the radiative equilibrium)

We are stating that absorbed energy $\int_0^\infty \kappa_\nu J_\nu \, d\nu =$ emitted energy $\int_0^\infty \kappa_\nu B_\nu \, d\nu$

This allows us to find a temperature structure $T(x)$:

– we begin with a given $\kappa_\nu(x), T(x), B_\nu(T(x)), \forall x$

– solve the transfer equation and obtain $J_\nu(x)$

– Is the equation $\int_0^\infty \kappa_\nu \left(J_\nu - B_\nu \right) d\nu = 0$ fulfilled? $\rightarrow$ No $\rightarrow$ correct $T(x)$; Yes $\rightarrow$ $T(x)$ correct

Energy equation – Radiative equilibrium

Assuming

- frequency-independent opacity
- LTE
- Eddington approximation (J= 3K through the whole atmosphere)

we obtain the gray temperature structure

(valid at the bottom of the atmosphere in radiative equilibrium)

$$T^4(\bar{\tau}) = \frac{3}{4} T_{\text{eff}}^4 \left(\bar{\tau} + \frac{2}{3} \right)$$

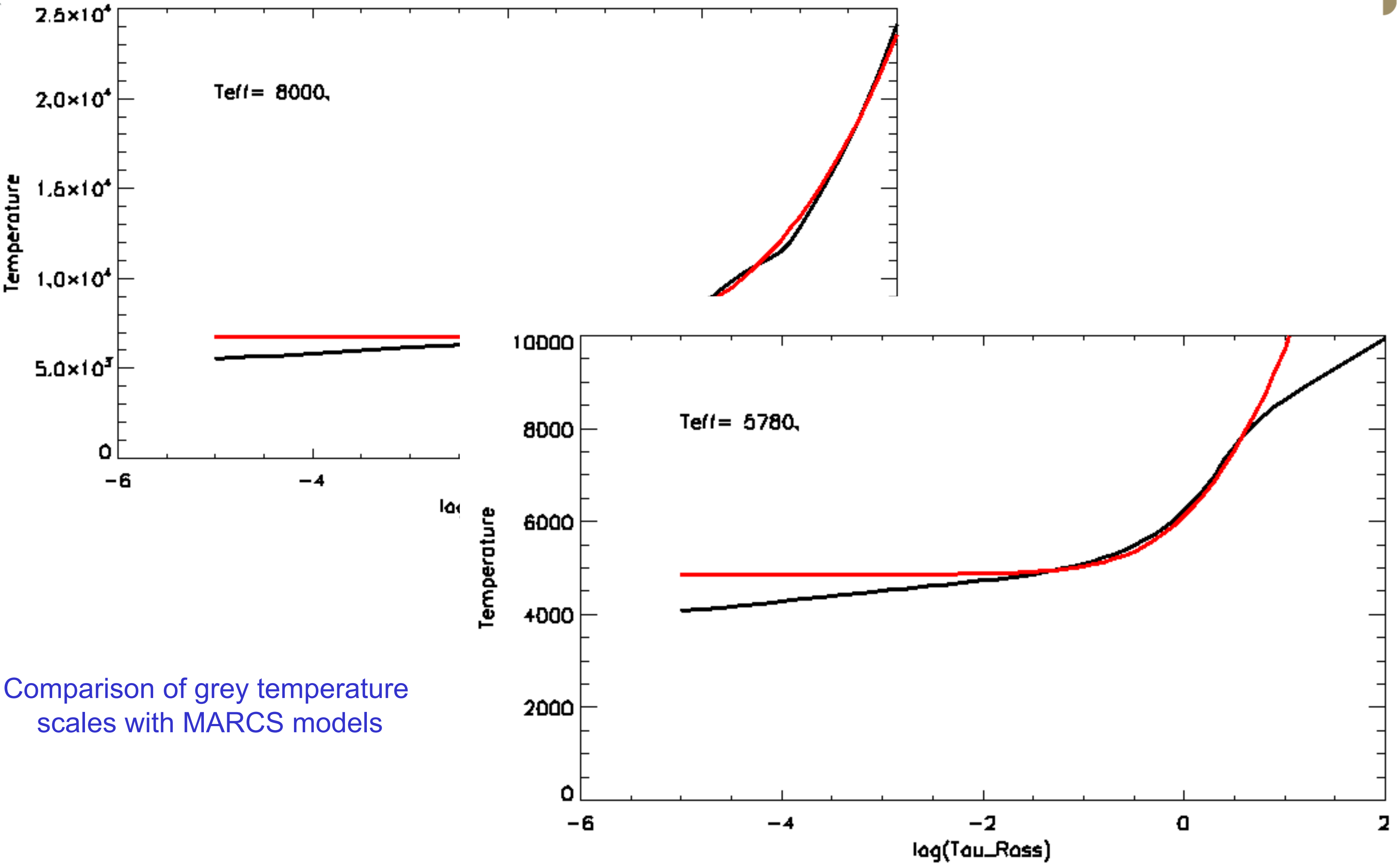
and without the Eddington approximation we get

$$T^4(\bar{\tau}) = \frac{3}{4} T_{\text{eff}}^4 \left(\bar{\tau} + q(\bar{\tau}) \right)$$

where $q(\bar{\tau})$ is a slowly varying function of $\bar{\tau}$, called the Hopf function. It varies between $1/\sqrt{3} = 0.57$ at $\bar{\tau}$=0 and 0.71 at $\bar{\tau}$=∞.

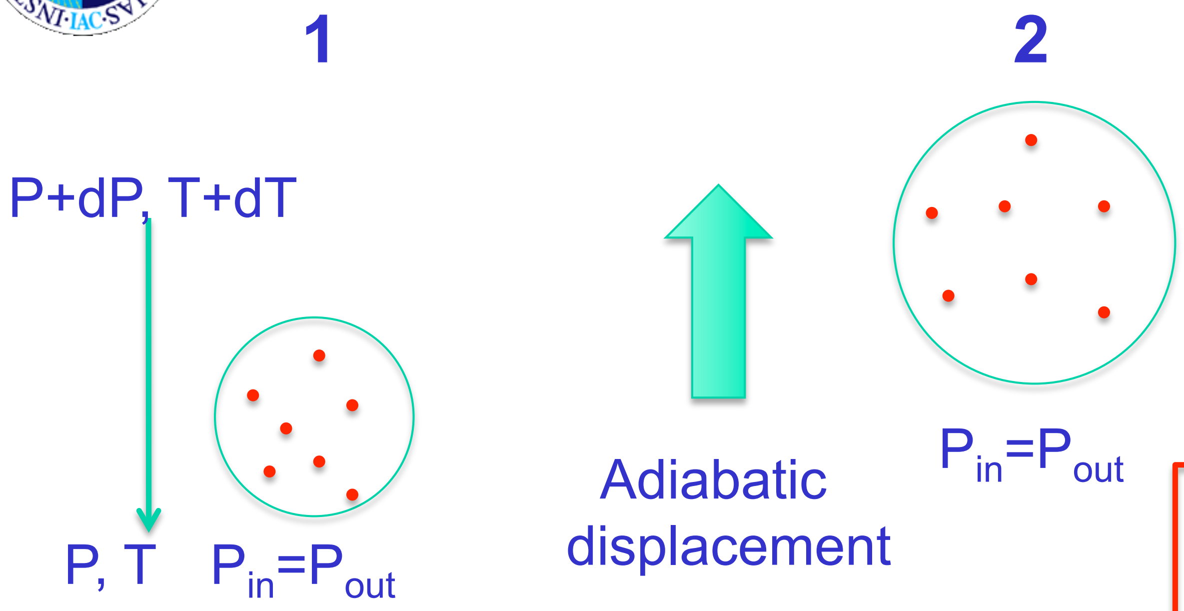
Note that $T(0) = 0.81 T_{\text{eff}}$ (0.84T_{eff} in the Eddington approximation) and $T\left(\bar{\tau} = 2/3\right) = T_{\text{eff}}$

Energy equation



Comparison of grey temperature scales with MARCS models

Energy equation - Convection



$\rho_{in} > \rho_{out} \Rightarrow$ gravity pulls down

$\Rightarrow$ convection supressed

$T_{in} < T_{out} \Rightarrow \left|\nabla T_{in}\right| > \left|\nabla T_{out}\right|$

$\Rightarrow \left|\nabla T_{ad}\right| > \left|\nabla T_{rad}\right|$ when no convection (stability)

Schwarzschild criterion for stability

$$\left|\frac{d\ln T}{d\ln P}\right|_{ad} > \left|\frac{d\ln T}{d\ln P}\right|_{rad}$$

Convection appears when the absolute adiabatic gradient is smaller than the radiative one

$$\left(\frac{dT(r)}{dr}\right)_{rad} = \frac{3}{4ac}\frac{\kappa}{T^3}\frac{L(r)}{4\pi r^2} \qquad \left(\frac{dT(r)}{dr}\right)_{ad} = \frac{\Gamma_2 - 1}{\Gamma_2}\frac{T(r)}{P(r)}\frac{dP(r)}{dr}$$

$$\frac{\Gamma_2 - 1}{\Gamma_2} = \left(\frac{d\ln P}{d\ln T}\right)_{ad} = \nabla_{ad} = -\frac{H_P}{T}\left(\frac{dT}{dr}\right)_{ad}$$

in a monoatomic, non-relativistic ideal gas

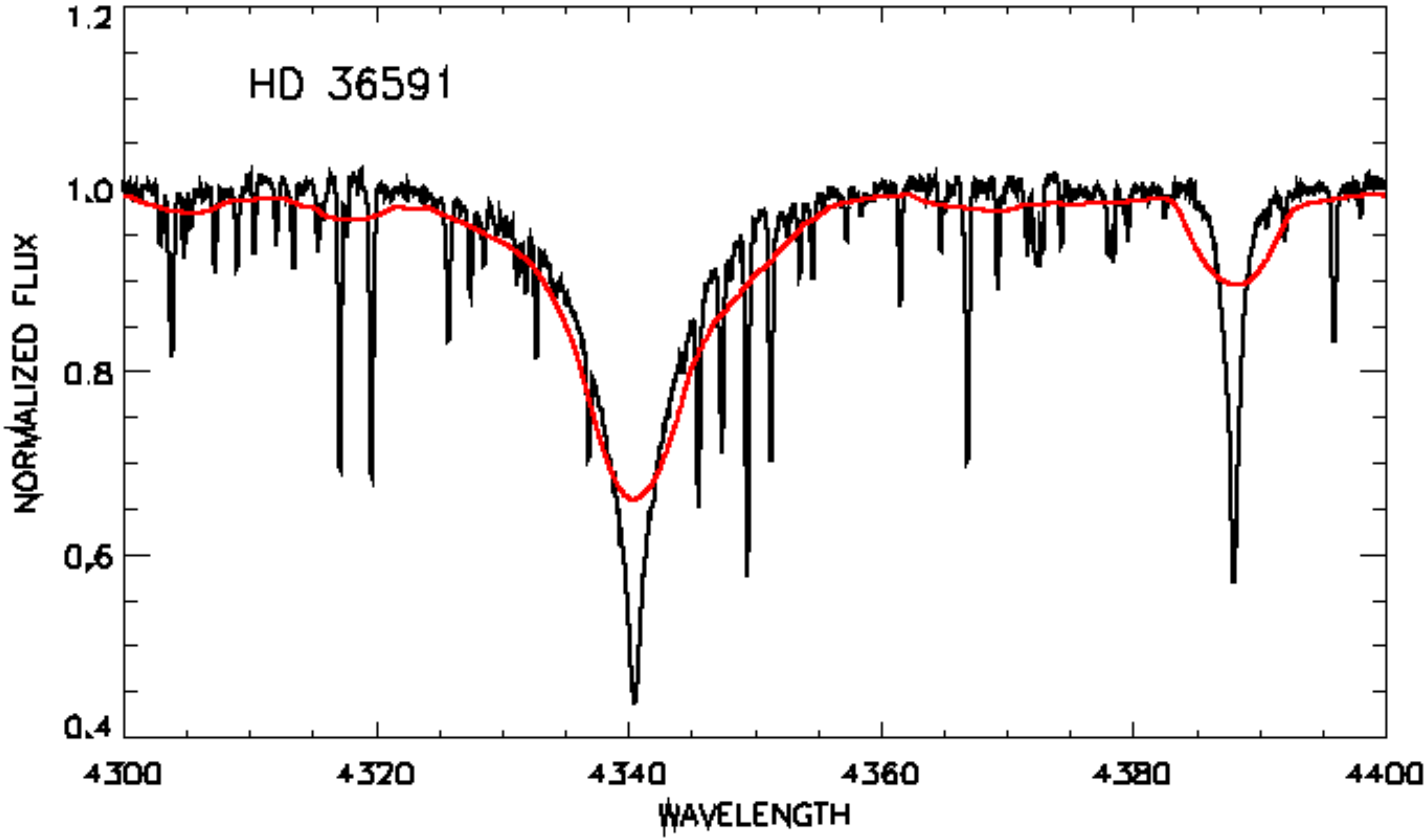
$\Gamma_2 = C_p / C_v = 5/3 \quad (\text{and } \Gamma_1 = \Gamma_2 = \Gamma_3 = \gamma)$

Favouring convection:

- large κ (kappa effect)
- $\Gamma_2 \sim 1$ (gamma effect)

This happens in ionization zones

Line Broadening- Intro



Line Broadening- Intro

There are two main causes of line broadening

1.- intrinsic: Heisenberg's uncertainty principle

$$\Delta E \Delta t \geq \frac{\hbar}{2} \Rightarrow \text{if } \Delta t \downarrow, \Delta E \uparrow$$

2.- extrinsic: Doppler effect

$$\lambda = \lambda_0 \left(1 \pm \frac{\upsilon}{c} \right)$$

and two main views

1.- Microscopic:

- affect the atomic absorption coefficient (for the atomic population)
- modify the total energy absorbed in the spectral line

2.- Macroscopic:

- do not affect the atomic absorption coefficient
- do not modify the total energy absorbed in the spectral line

Line Broadening- Intro

Origin / View	Intrinsic	Extrinsic
Microscopic	Natural Collisional	Thermal Microturbulence
Macroscopic		Rotation Macriturbulence

Natural line broadening

Natural broadening

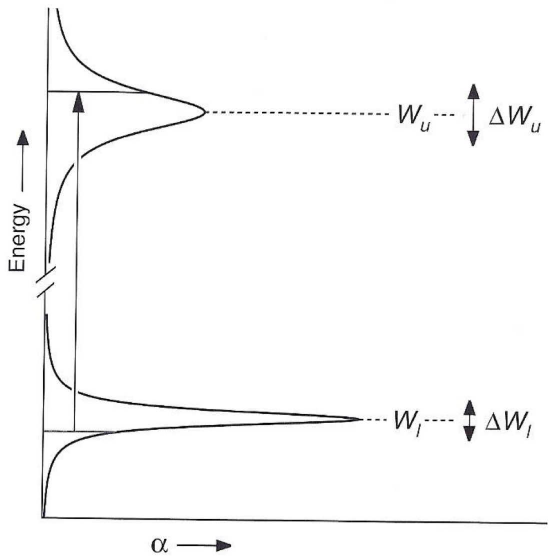
- Energy levels have a finite lifetime, and thus an energy uncertainty
- Natural broadening has a lorentzian profile

$$\alpha^{nat} = \frac{2\pi e^2}{mc}\frac{\gamma/2}{\Delta\omega^2+(\gamma/2)^2} = \frac{e^2}{mc}\frac{\gamma/4\pi}{\Delta\nu^2+(\gamma/4\pi)^2} = \frac{e^2}{mc}\frac{\lambda^2}{c}\frac{\gamma\lambda^2/4\pi c}{\Delta\lambda^2+(\gamma\lambda^2/4\pi c)^2}$$

with γ the radiative damping constant constante

For a l→u transition, γ_{ul} will be given by $\gamma_l + \gamma_u$

Damping constant for natural broadening



The emission (or absorption) profile of an *ul* (or *lu*) transition ia a lorentzian function with $\gamma = \gamma_u + \gamma_l$

If for a level u we also include induced emission and absorption (in addition to spontaneous emission):

$$\gamma_u = 4\pi\sum_{l<u} A_{ul} + 4\pi\sum_{l<u} I_\nu B_{ul} + 4\pi\sum_{k>u} I_\nu B_{uk}$$

Collisional line broadening

It is the perturbation of the energy levels of an atom caused by encounters (interaction) with electrons, ions or other atoms and molecules.

The resulting broadening will be proportional to the number of collisions, i.e., to particle density or pressure. Therefore, the collisional broadening is a good indicator of gravity

$$P(r) = P_0 e^{\frac{g\mu m_H}{k_B T}(r-r_0)}$$

- Perturbation will be larger for smaller inter-particle distance, R

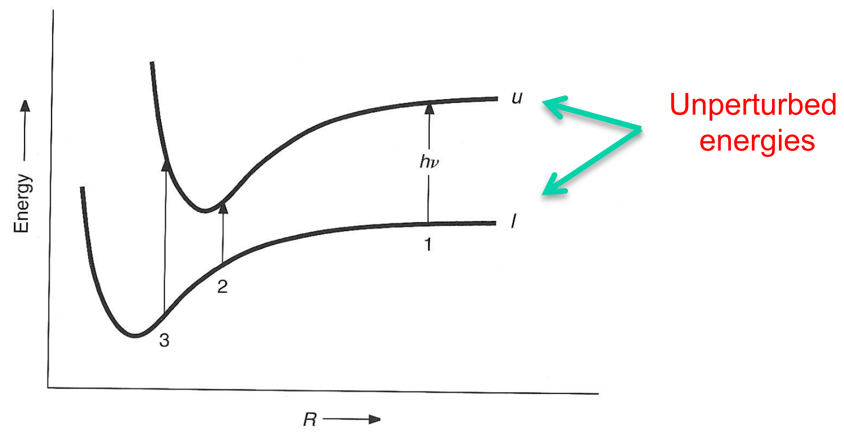


Fig. 11.2. The energies associated with the upper and lower atomic levels on a transition depend on the distance R to the perturber. The transition energy can be either less (2) or greater (3) than the unperturbed value (1).

The energy change can be expressed as a power law:

$$\Delta W = \text{constant}/R^n$$

where n depends on the kind of interaction

In frequencies:

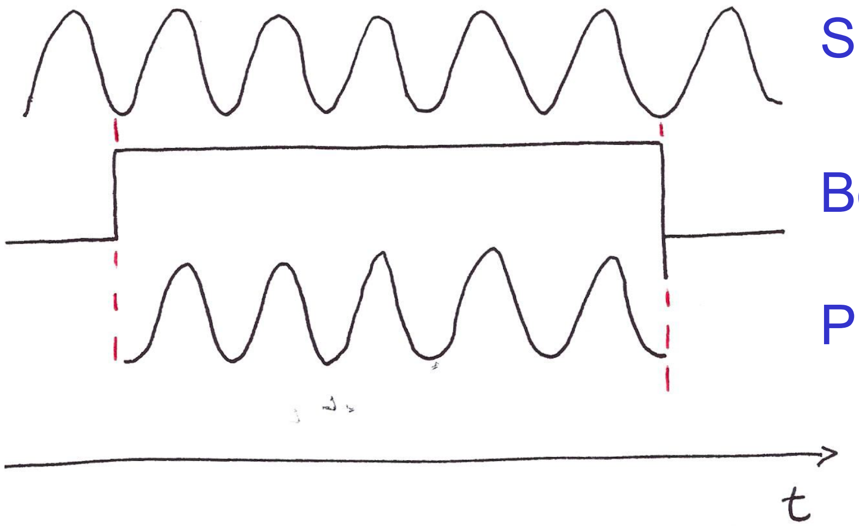
$$\Delta \nu = C_n / R^n$$

- C_n has to be calculated for each transition and kind of interaction

Collisional line broadening

The shape of the absorption profile is usually obtained in the impact approximation: the timespan of a collision is small compared to the time between collisions

- We assume that when an unperturbed atom emits a photon it behaves as an oscillating dipole during a time interval Δt. The resulting photon can be represented as the product of a sinusoid and a box function
  - The resulting frequency spectrum is a sinc function and its amplitude, $\mathrm{sinc}^2(\pi\Delta t\,(\nu-\nu_0))$, is centered at the sinusoid frequency with width $\Delta\nu = 1/\Delta t$



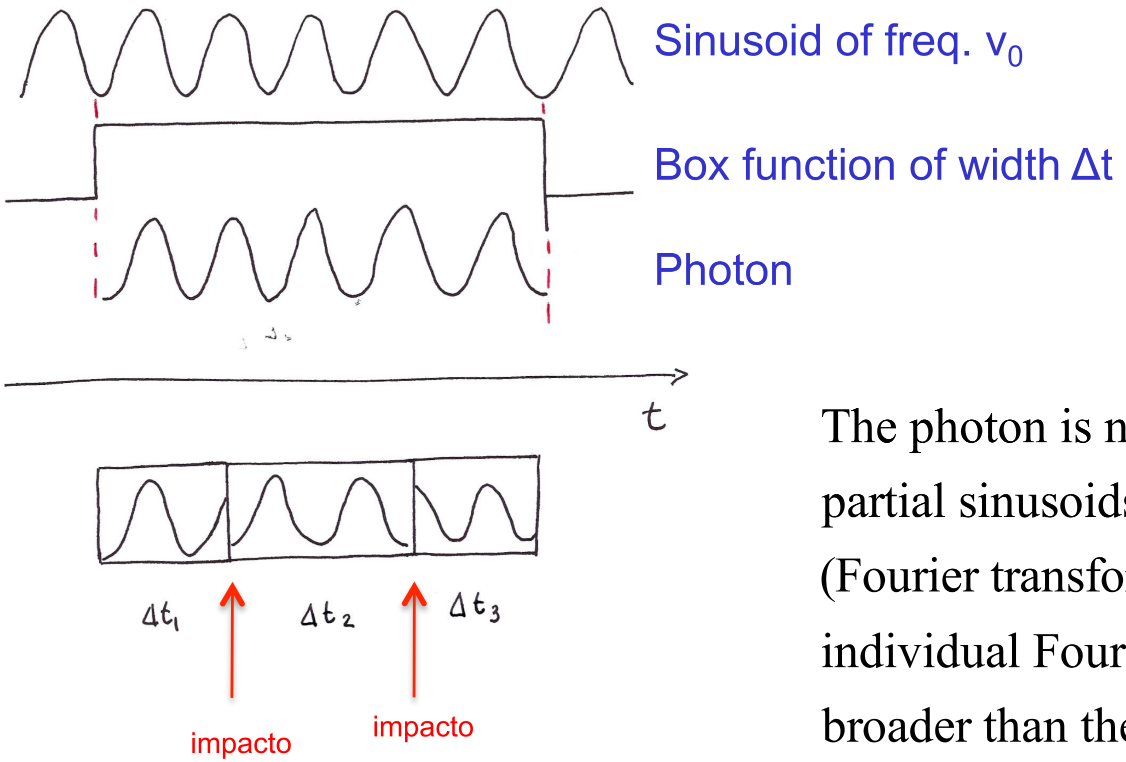
Sinusoid of freq. ν_0

Box function of width Δt

Photon

Collisional line broadening

- The perturbing particle produces a sudden phase shift in the sinusoid. The photon can be seen as the result of dividing the origibal wave in pieces, with a phase shift in each of them.



The photon is now the addition of the partial sinusoids. Its frequency spectrum (Fourier transform) is the sum of the individual Fourier transforms, each of them broader than the original one ($\Delta\nu = 1/\Delta t$), as $\Delta\nu_j = 1/\Delta t_j$

- The resulting profile will be a consequence of the Δt_j distribution, together with the sinc function of each of them

Collisional line broadening

The shape of the absorption coefficient will be proportional to the sinc^2 function corresponding to each Δt_j, weighted by the distribution of Δt_j and integrated to all possible Δt_j values

$$\alpha_\nu \propto \int_0^\infty \Delta t^2 \left(\frac{\sin \pi \Delta t(\nu - \nu_0)}{\pi \Delta t(\nu - \nu_0)} \right)^2 e^{-\Delta t / \Delta t_0} \frac{d\Delta t}{\Delta t_0}$$

which gives

$$\alpha_\nu = const \frac{\gamma_n / 4\pi}{(\nu - \nu_0)^2 + (\gamma_n / 4\pi)^2} \text{ with } \gamma_n = 2/\Delta t_0$$

Good news: that's a lorentzian, as the natural broadening

(the convolution of two lorentzian functions is a lorentzian function)

We need now the (collisional) damping constant, that will depend on the kind of interaction

Collisional line broadening

n	Kind	Lines affected	perturbers
2	Linear Stark	H, hydrogenic	p+ e-
3	Resonance	A-A	Same species
4	Quadratic Stark	Lines in hot stars	Ions, e-
6	Van der Waals	Lines in cool stars	H

Note: the linear Stark broadening is stronger than the quadratic one. The terms ‘linear’ and ‘quadratic’ refer to the first and second order in the Hamiltonian in perturbation theory

Some expressions (from Gray, 2002):

$$\log \gamma_4 \approx 19 + \frac{2}{3}\log C_4 + \log P_e - \frac{5}{6}\log T$$

$$\log \gamma_6 \approx 20 + 0.4\log C_6 + \log P_e - 0.7\log T$$

Collisional line broadening

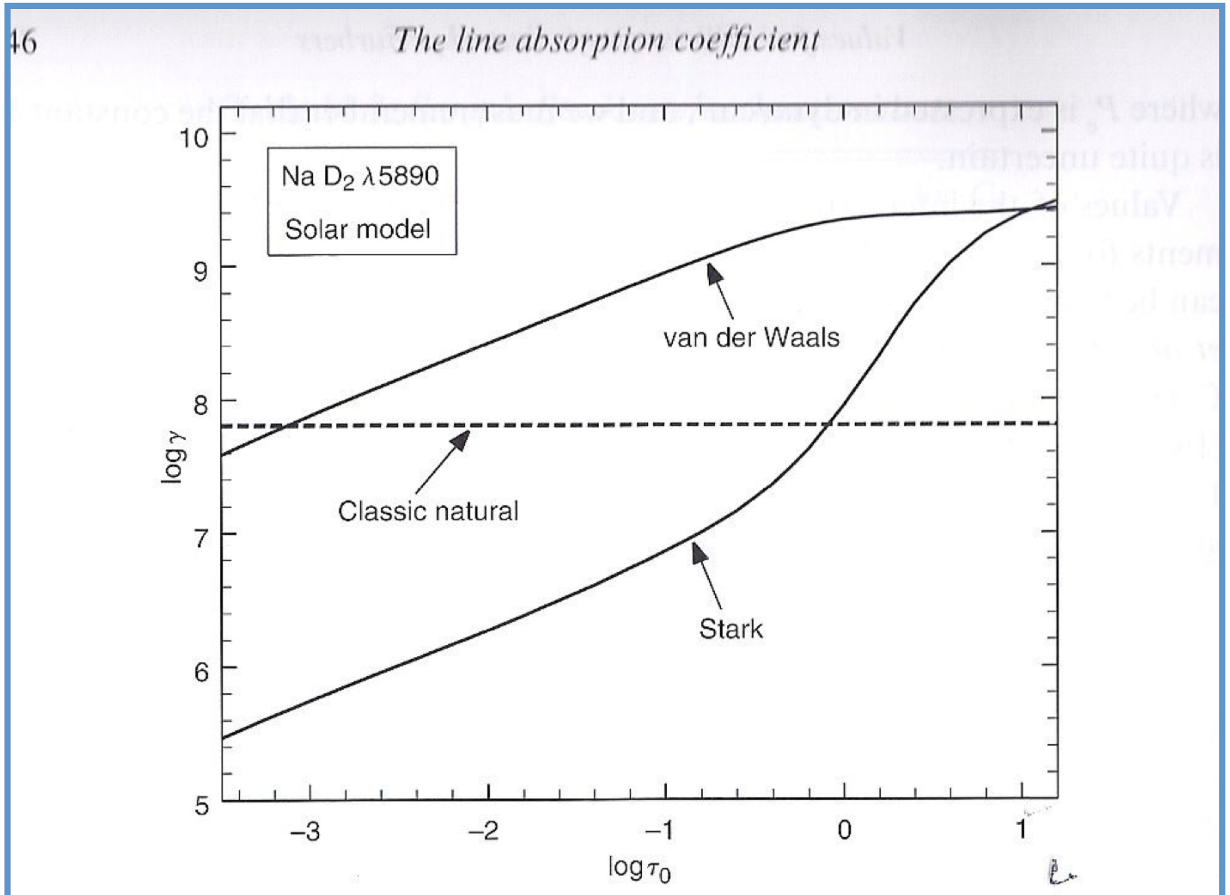


Fig. 11.4. Damping constants for the Na I D$_2$ line are shown as a function of depth in a solar model. The van der Waals damping constant is computed using Eq. (11.9). The Stark damping constant comes from Eq. (11.27). For omparison, the natural radiation damping according to Eq. (11.13) is shown.

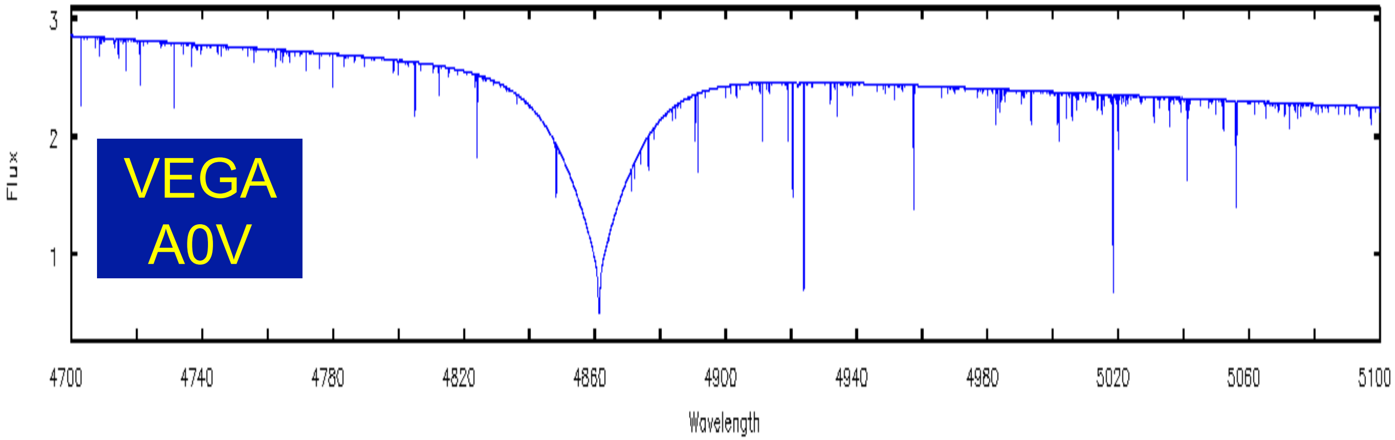
(Gray, 2002)

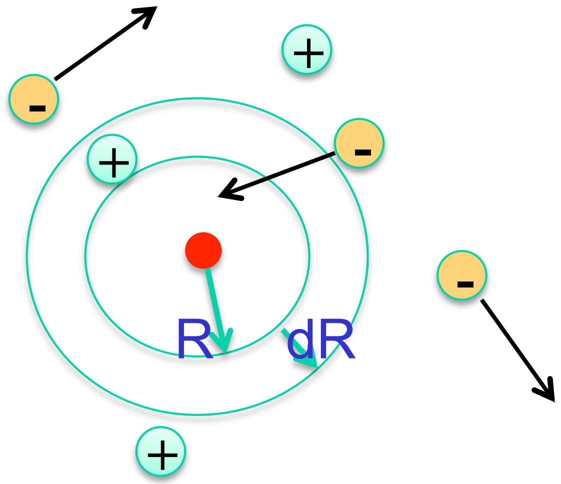
Collisional line broadening

The Stark brodening is caused by the electric field of a charged particle ($E= q/r^2$) on the perturbed atom

For H atoms, the linear Stark effect is particularly important when there are enough free electrons and ions (both because it is linear and because the nuclear charge is small).

We have to consider both the effect of the the slow protons (in the limit, the nearest neighbour approximation, although we shall consider the correct distribution of perturbers) and fast moving electrons (in the limit, the impact approximation)





Bad news: the result is a complicated expression that does not respond to an analytical standard expression (gaussian or lorentzian, f.e.). It can be tabulated.

Thermal line broadening

Atoms move in the atmosphere with velocity v_R (projected on the line of sight). This will produce a Doppler shift ($\Delta\lambda$) in the absorbed or emitted light

The $\Delta\lambda$ distribution will be proportional to the velocity distribution

The distribution of $\Delta\lambda$ will be

$$\frac{dN}{N} = \frac{1}{\pi^{1/2}\Delta\lambda_D} e^{-(\Delta\lambda/\Delta\lambda_D)^2} d\lambda$$

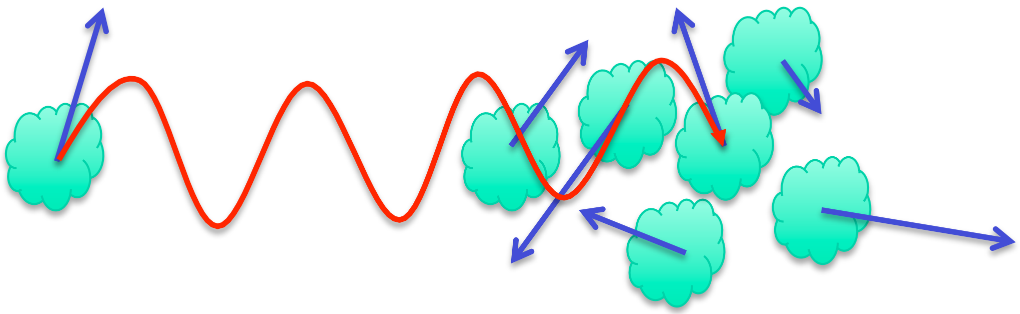
The total energy absorbed by the line will be $\frac{\lambda^2}{c}\frac{\pi e^2}{mc} f$ (in λ units), $\frac{\pi e^2}{mc} f$ (in ν units)

So that the absorption coefficient will be

$$\alpha d\lambda = \frac{\pi^{1/2} e^2}{mc} f \frac{\lambda_0^2}{c} \frac{1}{\Delta\lambda_D} e^{-(\Delta\lambda/\Delta\lambda_D)^2} d\lambda \qquad \alpha d\nu = \frac{\pi^{1/2} e^2}{mc} f \frac{1}{\Delta\nu_D} e^{-(\Delta\nu/\Delta\nu_D)^2} d\nu$$

The form of the line absorption coeffcient due to thermal broadening is a gaussian

Microturbulence



Microturbulence: absorbing elements of *smaller* size than the photon mean free path. They move in all directions with a gaussian velocity distribution. The absorbing atom sees the photon Doppler shifted.

As we assume a gaussian velocity distribution, microturbulence will have a gaussian profile (same as the thermal broadening)

It is an ad-hoc hypothesis to explain the broadening and strong absorption in some spectral lines

Hjerting and Voigt functions

The global coefficient will be the convolution of all individual coeffcients

$$\alpha(total) = \alpha(natural) * \alpha(colisional) * \alpha(termico) * \alpha(micro)$$

The natural and collisional broadenings will jointly give a lorentzian, and the thermal and microturbulence will give a gauusian. We will have:

$$\alpha_\nu = \frac{\pi e^2}{mc} f \frac{\gamma/4\pi^2}{\Delta\nu^2 + (\gamma/4\pi)^2} * \frac{1}{\pi^{1/2}\Delta\nu_D} e^{-(\Delta\nu/\Delta\nu_D)^2} = \frac{\pi^{1/2}e^2}{mc}\frac{f}{\Delta\nu_D} H(u,a) = \frac{\pi^{1/2}e^2}{mc}\frac{\lambda_0^2 f}{\Delta\lambda_D} H(u,a)$$

$H(u,a)$ is the Hjerting function, whose arguments are

$$u = \Delta\nu/\Delta\nu_D = \Delta\lambda/\Delta\lambda_D$$

$$a = \frac{\gamma}{4\pi\Delta\nu_D} = \frac{\gamma\lambda_0^2}{4\pi c\Delta\lambda_D}$$

and que

$$H(u,a) = \int_{-\infty}^{+\infty} \frac{\gamma/4\pi^2}{(\Delta\nu - \Delta\nu_1)^2 + (\gamma/4\pi)^2} e^{-(\Delta\nu_1/\Delta\nu_D)^2} d\nu_1 = \frac{a}{\pi}\int_{-\infty}^{+\infty}\frac{e^{-u_1^2}}{(u-u_1)^2 + a^2} du_1$$

Hjerting and Voigt functions

The Hjerting function can be expressed as a power expansion:

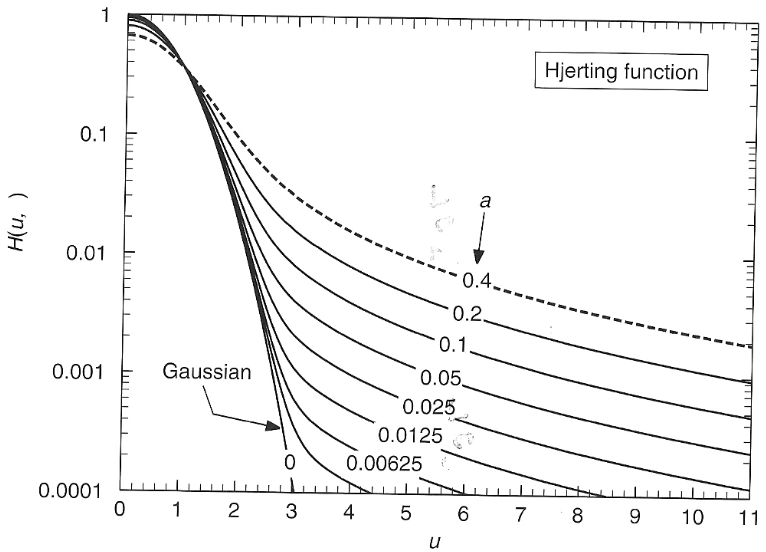
$$H(u,a) = H_0(u) + aH_1(u) + a^2 H_2(u) + a^3 H_3(u) + \ldots$$

where the H_j functions are tabulated. $H_0(u)$ is a gaussian and dominates the profile center.

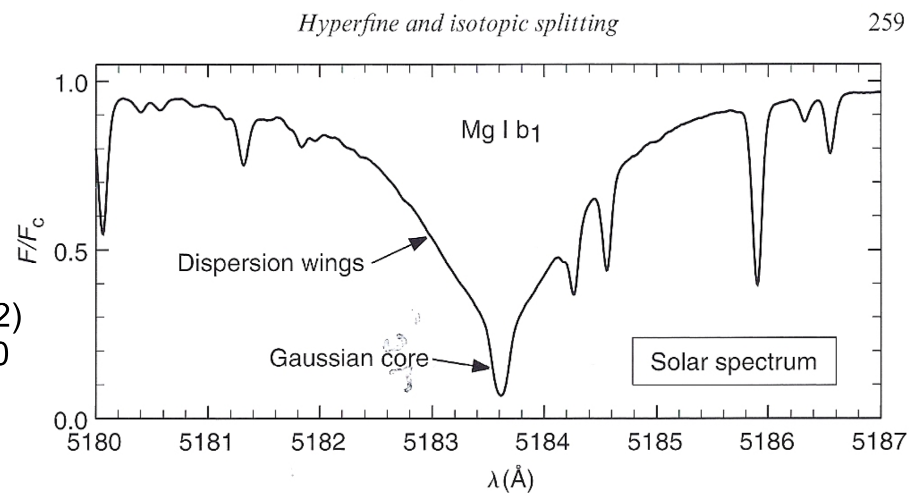
In the far wing the even terms in the expansion tend to zero and we can use (asymptotically)

$$H_1(u) = \frac{0.56419}{u^2} + \frac{0.846}{u^4} \quad \text{and} \quad H_3(u) = -\frac{0.56}{u^4}$$

It is customary to use the Voigt function $$V(u,a) = \frac{1}{\pi^{1/2}\Delta\nu_D} H(u,a)$$



Gray (2002) Figs. 11.10 and 11.11



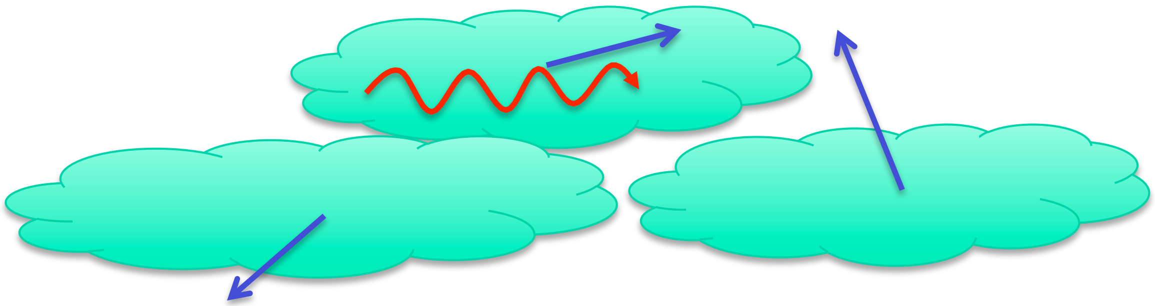
For Stark broadening (that does not result in a gaussian or lorentzian function) we have to perform a numerical convolution with the Hjerting or Voigt function

Line brodening- summary 1

Up to now we have seen processes that modify the atomic absorption coefficient

Process	Natural	Collisional (or Pressure)				Thermal	Microturbulence
		Resonance	van der Waals	Stark linear	Strak cuadrático		
Shape	Lorentz.	Lorentz.	Lorentz.			Gauss.	Gauss.
Physics	Heisenberg uncertainty principle	Collisions with identical atoms	Collisions with other atoms	Ions and e- electric fields on hydrogenic atoms	Ions and e- electric fields on non-hydrogenic atoms	Atoms thermal velocity distribution	Heuristic: gas cells of smaller size than the photon mean free path.
comment	Small broadening	Mainly collisions with H		Dominant in hot stars		Jointly with the lorentzian, results a Hjerting or Voigt profile	

Macroturbulence



Microturbulence: absorbing elements of *larger* size than the photon mean free path. The emitting and absorbing atoms are at rest w.r.t. each other.

Each macro cell behaves as an independent atmosphere. The emerging spectrum will be convolved with the velocity distribution of the cells

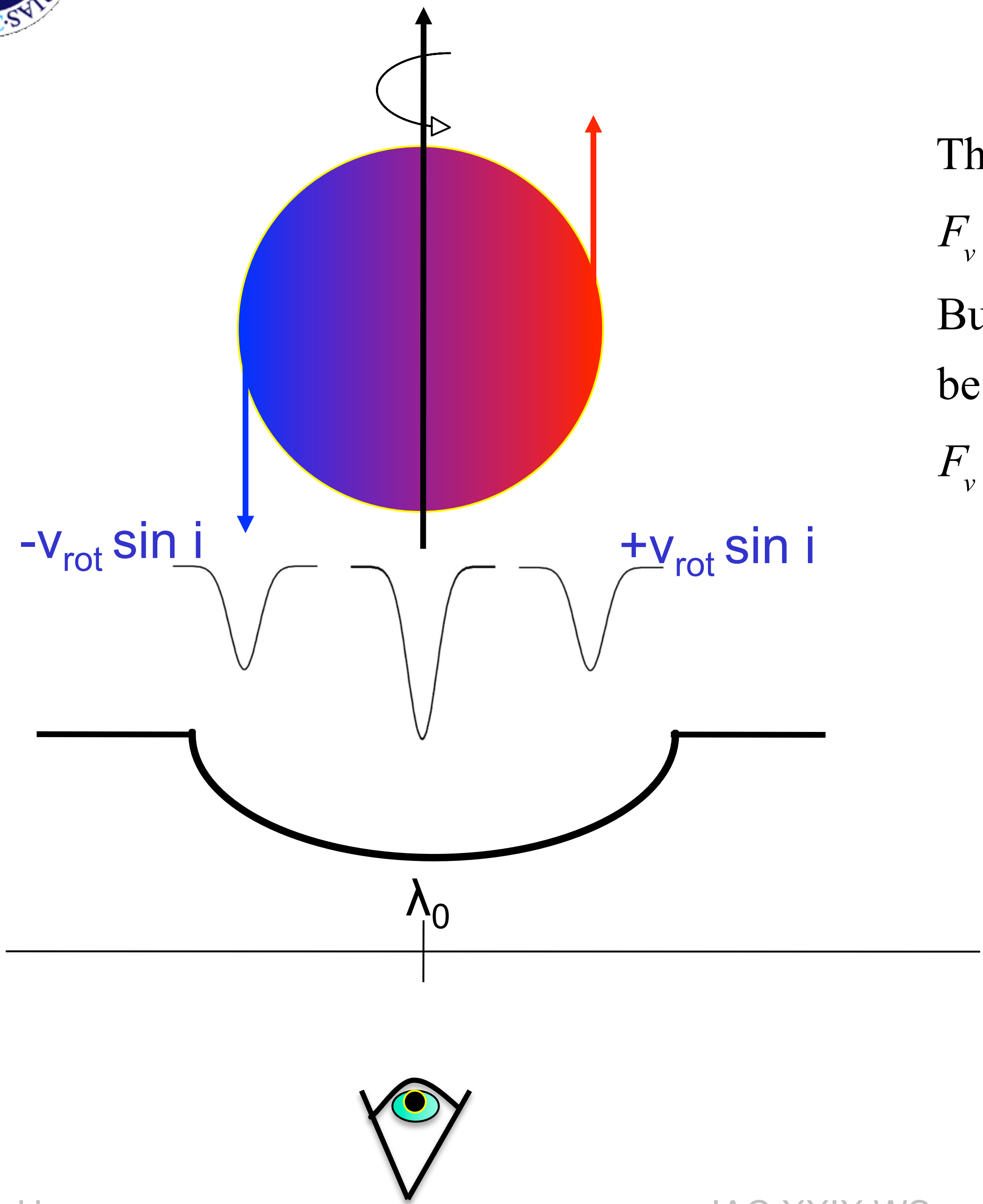
$$I_\nu = I_\nu^0 * \Theta(\Delta\lambda) \Rightarrow F_\nu = \oint I_\nu^0 * \Theta(\Delta\lambda)\cos\theta d\omega$$

A simple assumption is a gaussian velocity distribution, but it has been shown better to use a radial-tangential profile

$$\Theta(\Delta\lambda) = A_R\Theta_R(\Delta\lambda) + A_T\Theta_T(\Delta\lambda) = \frac{A_R}{\pi^{1/2}\zeta_R\cos\theta}e^{-(\Delta\lambda/\zeta_R\cos\theta)^2} + \frac{A_T}{\pi^{1/2}\zeta_T\sin\theta}e^{-(\Delta\lambda/\zeta_T\sin\theta)^2}$$

where ζ_R and ζ_T are the macroturbulent velocities in radial and tangential directions to the stellar surface, and θ is the angle of the line of sight with the normal to the surface

Rotational line broadening



The stellar emergent flux will still be

$$F_\nu = \oint I_\nu \cos\theta d\omega$$

But now the intensity at each point will be shifted by Doppler effect due to rotation

$$F_\nu = \oint I_\nu(\lambda - \Delta\lambda)\cos\theta\, d\omega$$

Rotational line broadening

This can again be expressed as a convolution with the rotation profile (that gives the distribution of $\Delta\lambda$)

$$F_\nu = \int_{-\infty}^{+\infty} I_\nu(\lambda - \Delta\lambda) G(\Delta\lambda) d\Delta\lambda = I_\nu(\lambda) * G(\lambda)$$

where

$$G(\Delta\lambda) = \frac{2(1-\varepsilon)\left[1-\left(\Delta\lambda/\Delta\lambda_L\right)^2\right]^{1/2} + \frac{1}{2}\pi\varepsilon\left[1-\left(\Delta\lambda/\Delta\lambda_L\right)^2\right]}{\pi\Delta\lambda_L\left(1-\varepsilon/3\right)} = c_1\left[1-\left(\frac{\Delta\lambda}{\Delta\lambda_L}\right)^2\right]^{1/2} + c_2\left[1-\left(\frac{\Delta\lambda}{\Delta\lambda_L}\right)^2\right]$$

and we have used the usual limb darkening law $I_c = I_c^0(1-\varepsilon+\varepsilon\cos\theta)$

Rotation and macroturbulence will be convolved numerically

Line brodening- summary 2

	Natural	Collisional	Thermal	Rotation	Micro	Macro
Origin	Uncertainty principle	Atomic interaction	Thermal velocity dispersion	Stellar rotation	Convection? (cool stars)	Convection (cool stars) Pulsation? (hot stars)
Shape	Lorentzian	Lorentzian (mostly)	Gaussian	own	Gaussian	Radial-tangential
Parameter	Aik, τ	Gravity	T (not in stars)	Rotational velocity	Abundances (needed for)	
Changes absorbed energy?	yes	yes	yes	no	yes	no
Objects	Atoms	Cool stars (van der Waals) Hot stars (Stark)	Low density hot plasmas	stars	stars	stars