NUMERICAL METHODS IN RADIATIVE TRANSFER

Part I

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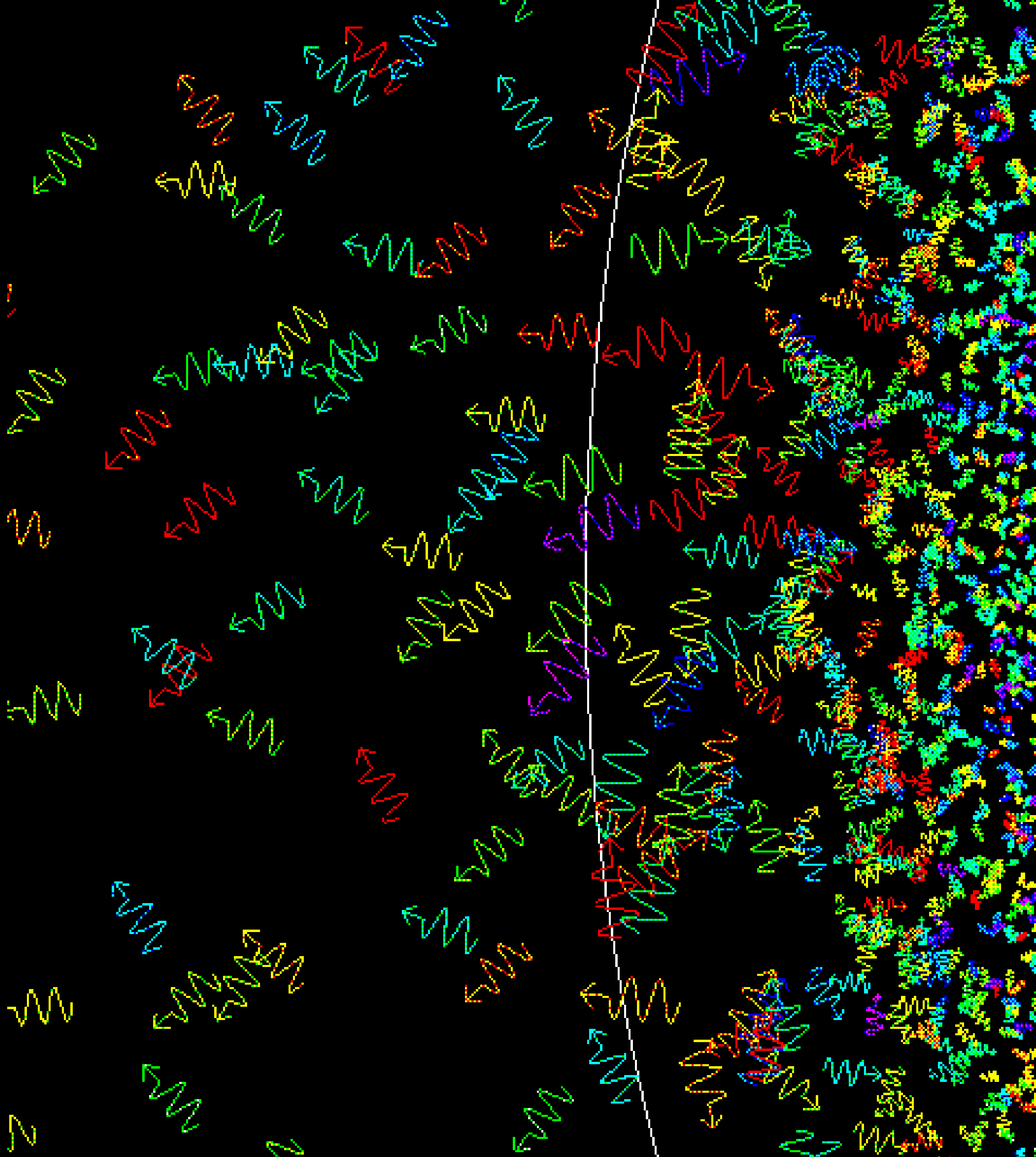
RADIATIVE TRANSFER



Radiative transfer (RT)

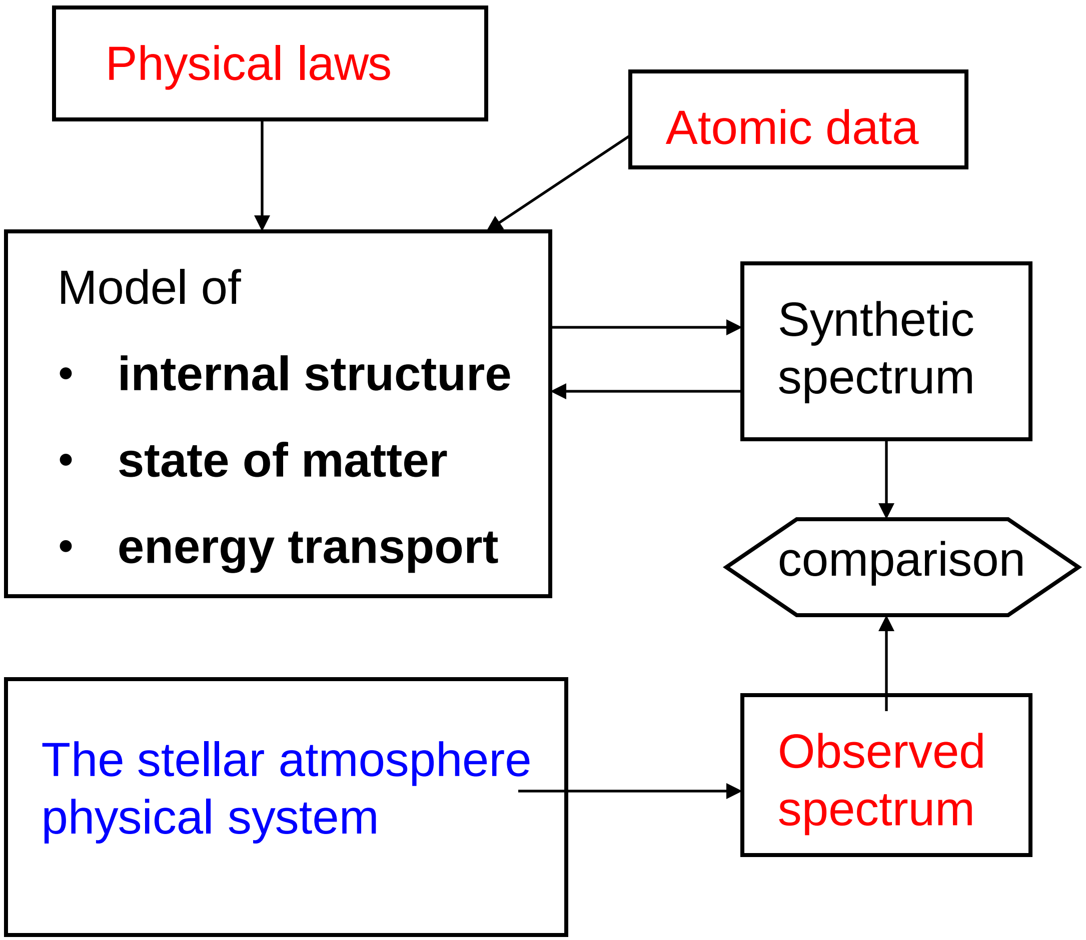
- **Radiative transfer** is a link between the macroscopic properties of celestial bodies (e.g. radiative flux they emit) and the microscopic interactions of photons with gas particles that determine the conditions on these objects.

- **Radiation** is the most important diagnostic tool we have at hands for the study of distant celestial objects.

- **Radiation** is not only a source of information; it also affects the physical state of the medium it propagates through

- **Radiative transfer** is a necessary step in the iterative process of astrophysical object modeling

Stellar atmospheres modeling



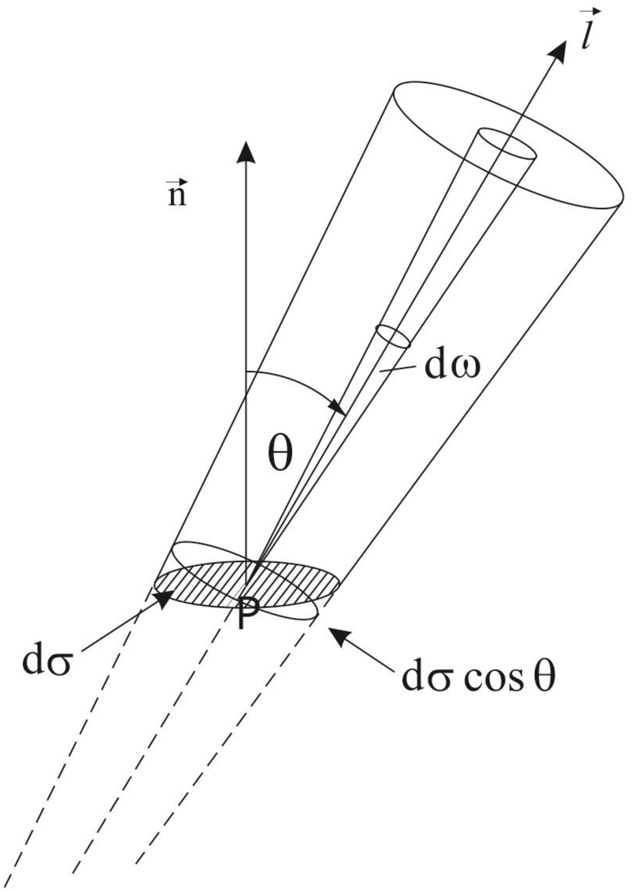
Outline

- I. The radiative transfer equation (RTE) and its formal solution
  (both in differential and integral form)

- II. Linear RT problems (two-level-atom line transfer)
  (direct and iterative methods)

- III. Non-linear RT problems (multi-level-atom line transfer)
  (iterative methods)

Specific intensity of radiation



$$dE_\nu = \boxed{I_\nu(\vec{r}, \vec{l}, t)}\, d\sigma \cos\theta\, d\nu\, d\omega\, dt$$

7 variables/parameters:

3 coordinates (Cartesian: *x,y,z*),
2 angles (θ, φ),
frequency (ν) and
time (*t*).

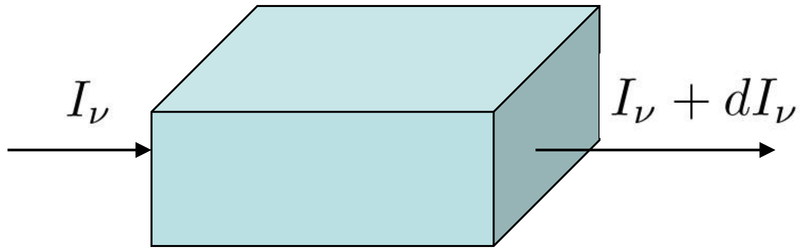
Radiative transfer equation (RTE)

$$dE_\nu^{\rm abs} = \boxed{\chi_\nu(\vec{r},\vec{l},t)} I_\nu(\vec{r},\vec{l},t)\,d\sigma\, ds\, d\omega\, d\nu\, dt$$

$$\chi_\nu = \chi_\nu^t + \chi_\nu^s$$

$$dE_\nu^{\rm em} = \boxed{\eta_\nu(\vec{r},\vec{l},t)}\, d\sigma\, ds\, d\omega\, d\nu\, dt$$

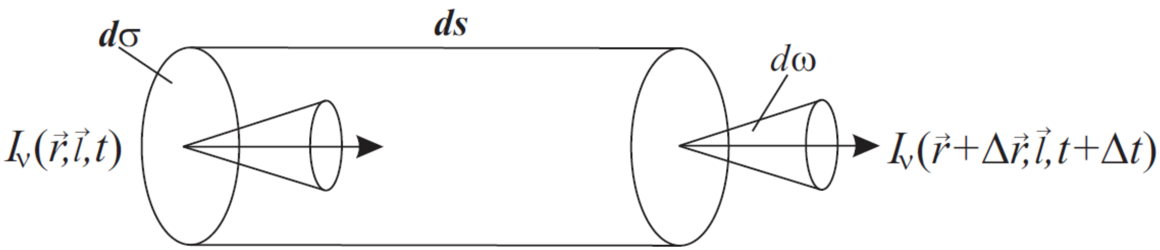
$$\eta_\nu = \eta_\nu^t + \eta_\nu^s$$



t – thermal processes

s – scattering processes

along a ray



$$[I_\nu(\vec{r}+d\vec{r},\vec{l},t+dt) - I_\nu(\vec{r},\vec{l},t)]\,d\sigma\, d\omega\, d\nu\, dt = dE_\nu^{\rm em} - dE_\nu^{\rm abs}$$

RTE: $$\boxed{[\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial s}] I_\nu(\vec{r},\vec{l},t) = \eta_\nu(\vec{r},\vec{l},t) - \chi_\nu(\vec{r},\vec{l},t) I_\nu(\vec{r},\vec{l},t)}$$

The most general form of the RTE

(that takes into account the variation of intensity with respect to all variables)

$$\frac{dI}{ds} = \frac{\partial I}{\partial t}\frac{\partial t}{\partial s} + \sum_{i=1}^{3}\frac{\partial I}{\partial r^i}\frac{\partial r^i}{\partial s} + \frac{\partial I}{\partial \Theta}\frac{\partial \Theta}{\partial s} + \frac{\partial I}{\partial \Phi}\frac{\partial \Phi}{\partial s} + \frac{\partial I}{\partial \nu}\frac{\partial \nu}{\partial s} =$$
$$\eta(\vec{r},\nu,\vec{l},t) - \chi(\vec{r},\nu,\vec{l},t)I(\vec{r},\nu,\vec{l},t)$$

7 variables/parameters:

3 coordinates (Cartesian:*x*,*y*,*z*),
2 angles (θ,φ),
frequency (ν) and
time (*t*).

Simplifications are needed!

$$\boxed{\left[\frac{1}{c}\frac{\partial}{\partial t} + (\vec{l}\cdot\nabla) + \frac{\partial\Theta}{\partial s}\frac{\partial}{\partial\Theta} + \frac{\partial\Phi}{\partial s}\frac{\partial}{\partial\Phi} + \frac{\partial\nu}{\partial s}\frac{\partial}{\partial\nu}\right] I(\vec{r},\nu,\vec{l},t) = \eta(\vec{r},\nu,\vec{l},t) - \chi(\vec{r},\nu,\vec{l},t) I(\vec{r},\nu,\vec{l},t)}$$

For the **time independent** case: $I_\nu \neq I_\nu(t) \rightarrow \boxed{I_\nu(\vec{r},\vec{l})}$

In a **static medium** - straight paths, constant frequency

$$\frac{\partial\Theta}{\partial s} = 0 \quad \frac{\partial\Phi}{\partial s} = 0 \quad \frac{\partial\nu}{\partial s} = 0$$

In **a moving medium** – two frames:

Laboratory (observer's) frame: LHS of RTE is simple, but transport coefficients are anisotropic

Comoving frame (CMF): LHS more complicated, transport coefficients are the same as in the static case

We will consider time-independent and static case:

$$\frac{\partial}{\partial t} = 0 \qquad \vec{v} = 0$$

In static media, transport coefficients are isotropic if the scattering is isotropic.

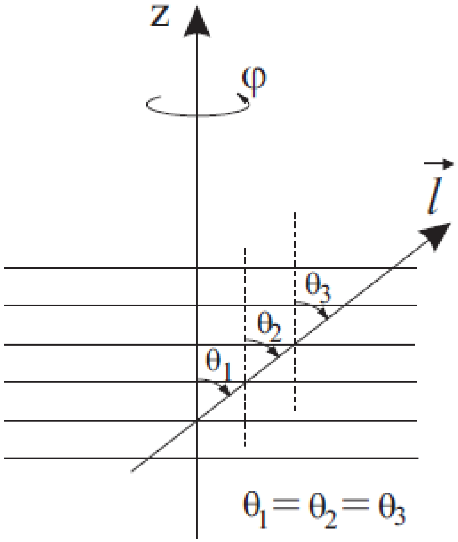
$$\boxed{\chi_\nu(\vec{r})} \quad \boxed{\eta_\nu(\vec{r})}$$

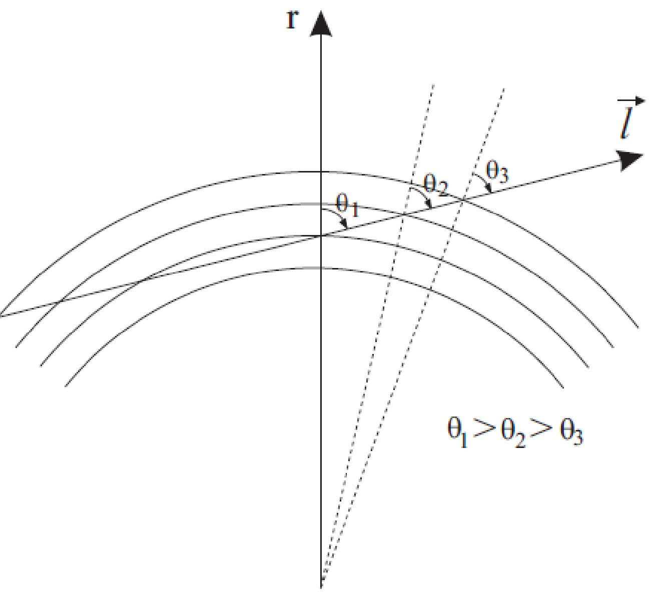
$$\eta^t_\nu(\vec{r}) = \chi^t_\nu(\vec{r}) \cdot B_\nu(T(\vec{r}))$$

$$\boxed{\eta^s_\nu(\vec{r},\vec{l})} = \sigma(\vec{r}) \int \frac{d\omega'}{4\pi} \int R(\nu',\vec{l}',\nu,\vec{l}) I_{\nu'}(\vec{r},\vec{l}) d\nu'$$

e.g. for coherent and isotropic scattering $\varphi(\nu')\delta(\nu-\nu')\boxed{g(\vec{l}',\vec{l})} = 1$

Geometry:





Assumption: atmosphere is stratified either in homogeneous plane-paralel or spherically symmetric layers.

Homogeneity = 1D ⟶ azimuthal symmetry

In **planar geometry**:

$$I_\nu(\vec{r},\vec{l}) \rightarrow \boxed{I_\nu(z,\theta)} \qquad \mu = \cos\theta$$

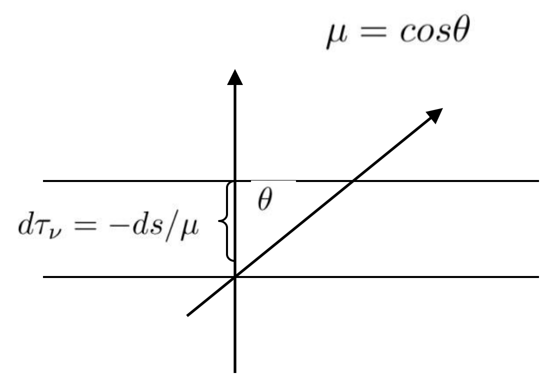
$$\boxed{\mu \frac{dI_\nu(z,\mu)}{dz} = \eta_\nu(z) - \chi_\nu(z) I_\nu(z,\mu)}$$

In **spherical geometry**:

$$I_\nu(\vec{r},\vec{l}) \rightarrow \boxed{I_\nu(r,\theta)}$$

$$\boxed{\mu \frac{\partial I_\nu(r,\mu)}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I_\nu(r,\mu)}{\partial \mu} = \eta_\nu(r) - \chi_\nu(r) I_\nu(r,\mu)}$$

Time-independent RT equation in the static, plane-parallel media with isotropic scattering :



RTE:

$$\boxed{\mu \frac{dI_\nu(z,\mu)}{dz} = \eta_\nu(z) - \chi_\nu(z) I_\nu(z,\mu)}$$

optical depth

$$\boxed{d\tau_\nu = -\chi_\nu(z) dz}$$

$$\tau_\nu(z) = \int_z^{z_{max}} \chi_\nu(z') dz'$$

source function

$$\boxed{S_\nu = \frac{\eta_\nu}{\chi_\nu}}$$

Is the number of mean free paths of a photon at frequency ν traveling from z_{max} to z

RTE:

$$\boxed{\mu \frac{dI_\nu(\tau_\nu,\mu)}{d\tau_\nu} = I_\nu(\tau_\nu,\mu) - S_\nu(\tau_\nu)}$$

Some simple 1D plane–parallel transfer problems

$$\boxed{\mu \frac{dI_\nu(z,\mu)}{dz} = \eta_\nu(z) - \chi_\nu(z) I_\nu(z,\mu)}$$

- no absorpion, no emission : $\chi_\nu = 0$ and $\eta_\nu = 0$

$$\mu \frac{dI_\nu(z,\mu)}{dz} = 0 \longrightarrow I_\nu = \mathrm{const}$$

- no absorption, only emission : $\chi_\nu = 0$ but $\eta_\nu \neq 0$

$$\mu \frac{dI_\nu(z,\mu)}{dz} = \eta_\nu(z) \longrightarrow I_\nu(z,\mu) = I_\nu(0,\mu) + \int_0^z \eta_\nu(z')dz'/\mu$$

- no emission, only absorption : $\eta_\nu = 0$ but $\chi_\nu \neq 0$

$$\mu \frac{dI_\nu(z,\mu)}{dz} = -\chi_\nu(z) I_\nu(z,\mu) \longrightarrow I_\nu(z,\mu) = I_\nu(0,\mu) e^{-\int_0^z \chi_\nu(z')dz'/\mu}$$

Radiation-matter interactions

$$\mu \frac{dI_\nu(\tau_\nu, \mu)}{d\tau_\nu} = I_\nu(\tau_\nu, \mu) - \boxed{S_\nu(\tau_\nu)}$$

TE $\longrightarrow$ LTE $\longrightarrow$ non-LTE

RTE :

$$\boxed{\mu \frac{dI_\nu(\tau_\nu,\mu)}{d\tau_\nu} = I_\nu(\tau_\nu,\mu) - S_\nu(\tau_\nu)}$$

- **TE**:

$$I_\nu = S_\nu = B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}$$

- **LTE**:

$$S_\nu = \frac{\eta_\nu}{\chi_\nu} = B_\nu(T), \qquad I_\nu \neq B_\nu(T)$$

$$\boxed{\mu \frac{dI_\nu(\tau_\nu,\mu)}{d\tau_\nu} = I_\nu(\tau_\nu,\mu) - B_\nu(T(\tau_\nu))}$$

NF x ND ordinary **differential** equations

- **non-LTE**:

example: **two-level atom** line formation

linear problem

direct or iterative methods of solution →

$$\mu \frac{dI_\nu(\tau_\nu,\mu)}{d\tau} = I_\nu(\tau_\nu,\mu) - S_\nu(\tau_\nu)$$

$$S(\tau_\nu) = \varepsilon B(\tau_\nu) + (1-\varepsilon)\int_{-\infty}^{\infty}\varphi_\nu d\nu \frac{1}{2}\int_{-1}^{1}\boxed{I_\nu(\tau_\nu,\mu)}d\mu$$

system of NF x ND **integro-differential** equations

Boundary conditions for the RTE

$$\mu \frac{dI_\nu(\tau_\nu,\mu)}{d\tau_\nu} = I_\nu(\tau_\nu,\mu) - S_\nu(\tau_\nu)$$

(A) Semi-infinite medium

- Upper boundary condition:

$$I_\nu^-(\tau_\nu = 0, \mu) = 0$$

(no incident radiation)

- Lower boundary condition:

$$lim_{\tau_\nu \to \infty} I_\nu(\tau_\nu,\mu) e^{-\tau_\nu/\mu} = 0$$

for analytical solutions

$$I_\nu(\tau_\nu,\mu) = S_\nu(\tau_\nu) + \mu \frac{dS_\nu}{d\tau_\nu}$$

for numerical solutions

diffusion approximation

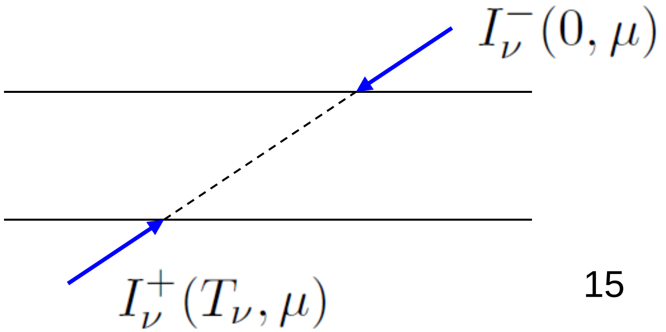
(at 1st order)

$$\rightarrow \boxed{\begin{array}{c} I_\nu^o = S_\nu \\ \mu dI_\nu^o/d\tau_\nu = I_\nu^1 - S_\nu \\ I_\nu^1 = S_\nu + \mu dS_\nu/d\tau_\nu \end{array}}$$

(B) Finite slab (spherical shell) of total optical thickness T

$$\tau_\nu = 0 : \quad I_\nu^-(0,\mu) = I_\nu(\tau_\nu = 0, \mu < 0)$$

$$\tau_\nu = T_\nu : I_\nu^+(T_\nu,\mu) = I_\nu(\tau_\nu = T_\nu, \mu > 0)$$



Formal solution

- The formal solution is a necessary step in any iterative solution of the full RT problem (self-consistent computation of the radiation field and the state of the medium). At each iteration step one solves the RTE with the known (from the previous iteration) source function, and uses the information provided by the formal solution within an efficient iterative procedure.

- It has to be performed many times $\Rightarrow$ it must be efficient !

- Methods that use the differential form of the RTE (e.g., Feautrier solution)

- Methods that use the integral form of the RTE (e.g., short characteristics approach)

Second-order differential form of the RTE

(along a ray subject to two-point boundary conditions)

Schuster (1905); generalized by Feautrier in 1964

The two-point boundary nature of the RT problem, i.e. the existence of two separate families of boundary conditions suggests a separate treatment of the in-going and out-going intensities (for two directions along the ray).

$$+\mu \frac{dI^+_{\nu\mu}}{d\tau_\nu} = I^+_{\nu\mu} - S_\nu$$

$$-\mu \frac{dI^-_{\nu\mu}}{d\tau_\nu} = I^-_{\nu\mu} - S_\nu$$

Feautrier's variables:

$$u_{\nu\mu} = \frac{1}{2}(I^+_{\nu\mu} + I^-_{\nu\mu})$$

$$v_{\nu\mu} = \frac{1}{2}(I^+_{\nu\mu} - I^-_{\nu\mu})$$

$$\mu \frac{dv_{\nu\mu}}{d\tau_\nu} = u_{\nu\mu} - S_\nu$$

$$\mu \frac{du_{\nu\mu}}{d\tau_\nu} = v_{\nu\mu}$$

$$\Rightarrow \quad \boxed{\mu^2 \frac{d^2 u_{\nu\mu}}{d\tau_\nu^2} = u_{\nu\mu} - S_\nu}$$

(for each frequency and direction)

Boundary conditions

$$\boxed{(\frac{du_{\nu\mu}}{d\tau}) = a + b\, u_{\nu\mu}}$$

at the surface $(\tau = 0)$ $\quad \mu(\frac{du_{\nu\mu}}{d\tau_\nu})_0 = u_{\nu\mu}(0) - I^-_{\nu\mu}(0)$

$$\boxed{\mu(\frac{du_{\nu\mu}}{d\tau_\nu})_0 = u_{\nu\mu}(0)} \qquad I^-_{\nu\mu}(\tau = 0) = 0$$

at $\tau = \tau_{max}$ $\quad \mu(\frac{du_{\nu\mu}}{d\tau_\nu})_{\tau_{max}} = I^+_{\nu\mu}(\tau_{max}) - u_{\nu\mu}(\tau_{max})$

$$I^+_{\nu\mu}(\tau_{max}) = B_\nu(\tau_{max}) + \mu(\frac{dB_\nu}{d\tau_\nu})_{\tau_{max}}$$

$$\boxed{\mu(\frac{du_{\nu\mu}}{d\tau_\nu})_{\tau_{max}} = \mu(\frac{dB_\nu}{d\tau_\nu})_{\tau_{max}}} \quad \text{or} \quad \boxed{(\frac{du_{\nu\mu}}{d\tau_\nu})_{\tau_{max}} = 0}$$

Feautrier solution

$$\boxed{\mu^2 (\frac{d^2 u}{d\tau^2})_l = u_l - S_l}$$

Discretization:

$\{\nu_i\}, (i = 1, NF)$
$\{\mu_j\}, (j = 1, ND)$
$\{\tau_l\}, l = 1, ..., N$

$$\boxed{u_{\nu_i, \mu_j}(\tau_l)}$$

$$\frac{\mu^2}{\Delta\tau_{l-1/2}\Delta\tau_l} u_{l-1} - \frac{\mu^2}{\Delta\tau_l}\left(\frac{1}{\Delta\tau_{l-1/2}} + \frac{1}{\Delta\tau_{l-1/2}}\right) u_l + \frac{\mu^2}{\Delta\tau_{l+1/2}\Delta\tau_l} u_{l+1} = u_l - S_l$$

$$B_1 u_1 + C_1 u_2 = L_1$$

$$A_l\, u_{l-1} - B_l\, u_l + C_l\, u_{l+1} = L_l$$

$$A_N u_{N-1} + B_N u_N = L_N$$

$$\begin{pmatrix} B_1 & C_1 & & & & & \\ -A_2 & B_2 & -C_2 & & & & \\ & -A_3 & B_3 & -C_3 & & & \\ & & . & . & . & & \\ & & & . & . & . & \\ & & & & -A_{N-1} & B_{N-1} & -C_{N-1} \\ & & & & & -A_N & B_N \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ . \\ . \\ u_{N-1} \\ u_N \end{pmatrix} = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \\ . \\ . \\ L_{N-1} \\ L_N \end{pmatrix}$$

or $\boxed{\hat{T}\,\vec{u} = \vec{L}}$

Forward elimination/back substitution

Forward elimination: derive for each layer (*l*,*l+1*) the coefficients of the recursive relation

$$\boxed{u_l = q_l + D_l\, u_{l+1}}$$

$$D_l = -(B_l + A_l D_{l-1})^{-1} C_l$$
$$q_l = (B_l + A_l D_{l-1})^{-1} (L_l - A_l q_{l-1})$$

At the top: $A_1=0$

At the bottom $C_N = 0 \longrightarrow D_N = 0 \longrightarrow u_N = q_N$

then back-substitution $u_l = q_l + D_l\, u_{l+1}$ up to the surface $l=1$.

Two-point algorithm

$$\boxed{\mu^2 (\frac{d^2 u}{d\tau^2})_l = u_l - S_l}$$ **RTE**

Upper boundary condition: $(\frac{du}{d\tau})_1 = \boxed{a_1} + \boxed{b_1} u_1$

$$(\frac{du}{d\tau})_{l-1} = a_{l-1} + b_{l-1}\, u_{l-1} \qquad (l-1, l)$$

The relation between u_{l-1} and u_l is obtained from:

$$u_l = u_{l-1} + \Delta\tau \left(\frac{du}{d\tau}\right)_{l-1} + \frac{1}{2}\Delta\tau^2 \left(\frac{d^2 u}{d\tau^2}\right)_{l-1}$$ Taylor's expansion

$$\Rightarrow \quad u_{l-1} = \boxed{q_{l-1}} + \boxed{D_{l-1}} u_l$$ recursive relation

The coefficients of boundary condition for the next layer are found from

$$\left(\frac{du}{d\tau}\right)_l = \left(\frac{du}{d\tau}\right)_{l-1} + \Delta\tau \left(\frac{d^2 u}{d\tau^2}\right)_{l-1} + \frac{1}{2}\Delta\tau^2 \left(\frac{d^3 u}{d\tau^3}\right)_{l-1}$$ Taylor's expansion

$$\left(\frac{d^3 u}{d\tau^3}\right)_{l-1} = \left(\frac{d^3 u}{d\tau^3}\right)_l = \frac{1}{\Delta\tau}\left[\left(\frac{d^2 u}{d\tau^2}\right)_l - \left(\frac{d^2 u}{d\tau^2}\right)_{l-1}\right]$$

$$\Rightarrow \quad (\frac{du}{d\tau})_l = \boxed{a_l} + \boxed{b_l} u_l$$

$$a_N + b_N u_N = \boxed{a_N^*} + \boxed{b_N^*} u_N$$

Formal solution of the RTE in integral form

$$\boxed{\mu \frac{dI_\nu(\tau_\nu, \mu)}{d\tau_\nu} = I_\nu(\tau_\nu, \mu) - S_\nu(\tau_\nu)}$$

$$\mu \frac{\partial}{\partial \tau_\nu}[I_{\nu,\mu}(\tau_\nu) e^{-\tau_\nu/\mu}] = e^{-\tau_\nu/\mu} S_\nu(\tau_\nu)$$

integrating factor

$$\boxed{I_{\nu,\mu}(\tau_1) = I_{\nu,\mu}(\tau_2) e^{-(\tau_2-\tau_1)/\mu} + \int_{\tau_1}^{\tau_2} S_\nu(t_\nu) e^{-(t_\nu-\tau_1)/\mu} dt_\nu/\mu}$$

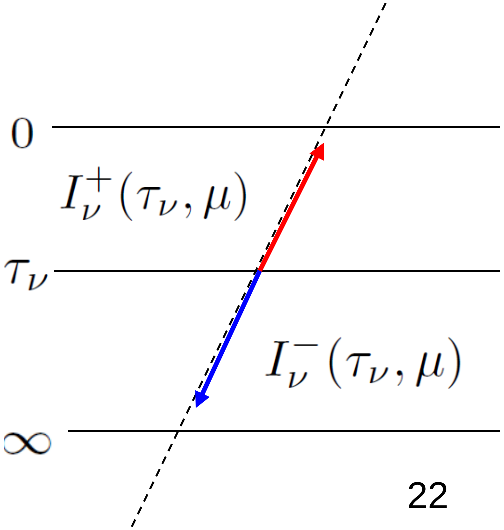
The general formal solution of the RTE for a semi-infinite plane-parallel atmosphere:

- Inward intensity (using the upper boundary condition)

$$I_\nu^-(\tau_\nu, \mu) = -\int_0^{\tau_\nu} S_\nu(t_\nu) e^{-(t_\nu-\tau_\nu)/\mu} \frac{dt_\nu}{\mu}$$

- Outward intensity (using the lower boundary condition)

$$I_\nu^+(\tau_\nu, \mu) = \int_{\tau_\nu}^{\infty} S_\nu(t_\nu) e^{-(t_\nu-\tau_\nu)/\mu} \frac{dt_\nu}{\mu}$$



Some simple 1D plane–parallel transfer problems

$$\mu \frac{dI_\nu(\tau_\nu,\mu)}{d\tau_\nu} = I_\nu(\tau_\nu,\mu) - S_\nu(\tau_\nu)$$

- absorption and emission : $\chi_\nu \neq 0$ and $\eta_\nu \neq 0$

$$\boxed{I(\tau_1,\mu) = I(\tau_2,\mu)e^{-(\tau_2-\tau_1)/\mu} + \int_{\tau_1}^{\tau_2} S(t)e^{-(t-\tau_1)/\mu}\frac{dt}{\mu}}$$

- semi-infinite atmosphere : $\tau_1 = 0 \quad \tau_2 = \infty$

$$\boxed{I_\nu(0,\mu) = \int_0^\infty S_\nu(t_\nu)e^{-t_\nu/\mu}\frac{dt_\nu}{\mu}}$$

- finite, homogeneous slab : $I_\nu(0,1) = (1-e^{-T_\nu})S_\nu$

$T_\nu >> 1 \quad I_\nu(0,1) = S_\nu$ $\qquad T_\nu << 1 \quad I_\nu(0,1) \to S_\nu T_\nu$

Moments of specific intensity in 1-D (in Eddington's notation)

$$\mu = \cos\theta$$

The zeroth angular moment
- **Mean intensity**

$$J_\nu(z) = \frac{1}{2}\int_{-1}^{1} I_\nu(z,\mu)d\mu$$

The first angular moment -
Eddington flux

$$H_\nu(z) = \frac{1}{2}\int_{-1}^{1} I_\nu(z,\mu)\mu d\mu$$

The second angular moment
- **K-integral**

$$K_\nu(z) = \frac{1}{2}\int_{-1}^{1} I_\nu(z,\mu)\mu^2 d\mu$$

Mean intensity

$$J_\nu(\tau_\nu) = \frac{1}{2}\int_{-1}^{1} I_\nu(\tau_\nu,\mu)d\mu = \frac{1}{2}\int_0^\infty S_\nu(t_\nu)E_1|t_\nu - \tau_\nu|dt_\nu = \Lambda_\tau\{S_\nu(t_\nu)\}$$

Λ operator

$$\Lambda_\tau\{...\} = \frac{1}{2}\int_0^\infty \{...\}E_1|t-\tau|dt$$

$$E_n(x) = \int_1^\infty \frac{e^{-xy}}{y^n}dy = \int_0^1 e^{-x/\mu}\mu^{n-1}\frac{d\mu}{\mu}$$

$$\Rightarrow \quad \boxed{J_\nu(\tau_\nu) = \Lambda_\tau\{S_\nu(t_\nu)\}}$$

Flux

$$F_\nu(\tau_\nu) = 2\int_{-1}^{1} I_\nu(\tau_\nu,\mu)\mu d\mu =$$

$$= 2\int_{\tau_\nu}^{\infty} S_\nu(t_\nu)E_2(t_\nu - \tau_\nu)dt_\nu - 2\int_0^{\tau_\nu} S_\nu(t_\nu)E_2(\tau_\nu - t_\nu)dt_\nu$$

K-integral

$$K_\nu(\tau_\nu) = \frac{1}{2}\int_{-1}^{1} I_\nu(\tau_\nu,\mu)\mu^2 d\mu =$$

$$= \frac{1}{2}\int_0^{\infty} S_\nu(t_\nu)E_3|t_\nu - \tau_\nu|dt_\nu$$

The essence of the whole story

The emergent intensity:

$$I_\nu^+(0,\mu) = \int_0^\infty S_\nu(t_\nu) e^{-t_\nu/\mu} \frac{dt_\nu}{\mu}$$

The emergent radiative flux:

$$\boxed{F_\nu^+(0)} = 2\int_0^1 I_\nu^+(0,\mu)\mu d\mu = 2\int_0^\infty \boxed{S_\nu(t_\nu)} E_2(t_\nu) dt_\nu$$

$$\downarrow$$

$$S_\nu(\tau_\nu) = \frac{\eta_\nu(\tau_\nu)}{\chi_\nu(\tau_\nu)}$$

Radiative transfer puts into relation the **radiative flux**, one of the fundamental stellar parameters, with the **microscopic interaction processes**.

RTE in integral form

$$I(\tau_L, \mu) = I(\tau_U, \mu)e^{-(\tau_U - \tau_L)} + \int_{\tau_L}^{\tau_U} S(t)e^{-(t-\tau_L)}dt$$

For the numerical integration we may either assume that the behavior of the source function between any pair of its known values at two points can be approximated by a polynomial or replace the integral by a quadrature sum.

Long characteristics (full polynomial interpolation) : *U* is the point at the boundary at which the ray penetrates into the medium; a single function is fitted to the full set of given data (global fits)

$$f(x) = \sum_{k=1}^{N} \left(\prod_{j=1, j \neq k}^{N} \frac{x - x_j}{x_k - x_j} \right) y_k$$

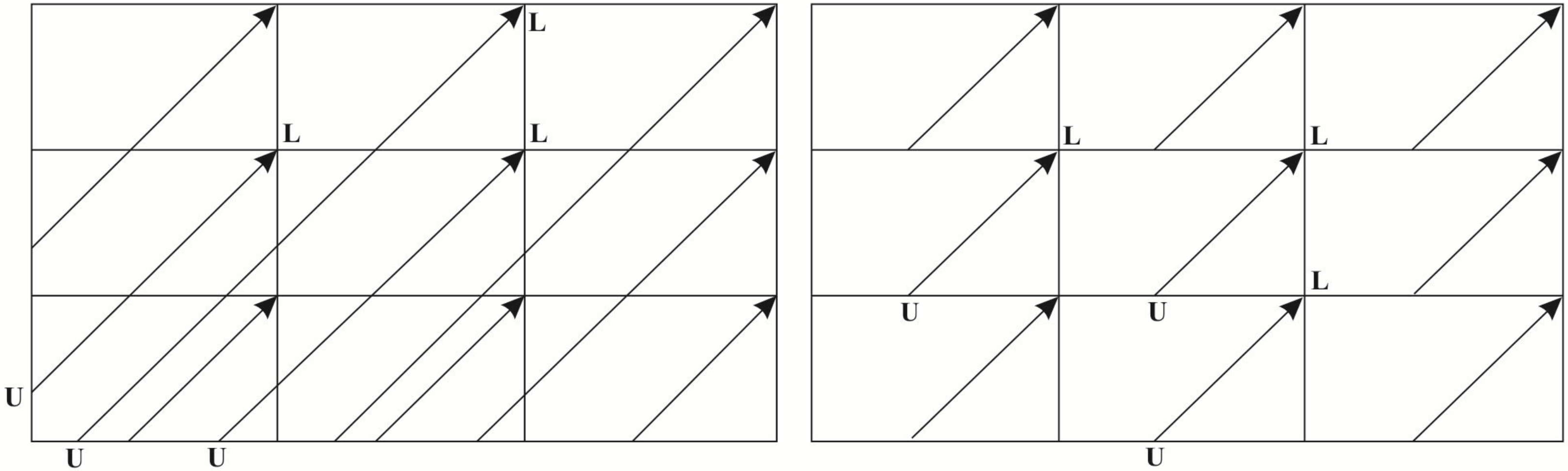
Lagrange form of the polynomial that goes through *N* points

Short characteristics (piece-wise interpolation) - ray segments (local fits)

$$\int_{\tau_0}^{\tau_{max}} S(t)e^{-(t-\tau_0)}dt = \sum_{l=1}^{l=N} \int_{\tau_l}^{\tau_{l+1}} S(t)e^{-(t-\tau_l)}dt$$

D.Mihalas, L.Auer & B.Mihalas (1978)

Long vs. short characteristics in 2D (Auer, 2003)



Long Characteristics Short Characteristics

Short characteristics

The general formal solution can be applied to the interval between two successive grid points

$$I^+(\tau_{l-1}) = I^+(\tau_l)e^{-\Delta} + \int_{\tau_{l-1}}^{\tau_l} S(t)e^{-(t-\tau_{l-1})}dt \qquad \text{for } \mu \geq 0$$

$$I^-(\tau_l) = I^-(\tau_{l-1})e^{-\Delta} + \int_{\tau_{l-1}}^{\tau_l} S(t)e^{-(\tau_l-t)}dt \qquad \text{for } \mu \leq 0$$

The integrals can be solved by assuming some polynomial representation for *S*

1st order (linear) short characteristics:

$$S(t) = S_l\frac{t-\tau_{l-1}}{\Delta\tau_l} + S_{l-1}\frac{\tau_l - t}{\Delta\tau_l} \qquad \Delta = \Delta\tau_l/|\mu|$$

$$\Delta\tau_l = \tau_l - \tau_{l-1}$$

$$\Rightarrow \quad I^+(\tau_{l-1}) = I^+(\tau_l)e^{-\Delta} + p_l S_{l-1} + q_l S_l$$

$$p_l = 1 - \frac{1-e^{-\Delta}}{\Delta} \qquad q_l = \frac{1-e^{-\Delta}}{\Delta} - e^{-\Delta}$$

Short characteristics (Kunasz and Auer, 1987)

Formal solution based on the SC solution of the first-order differential RT equation and parabolic approximation for the source function in three points

Widely exploited within the ALI (Approximate Lambda Iteration) methods

The integral $$\int_{\tau_U}^{\tau_L} S(t) e^{t-\tau_U}\, dt$$

is expressed in terms of the source function in ***three points***:
U (**upwind**), **L** (**local**) and **D** (**downwind**)

$$\int_{\tau_U}^{\tau_L} S(t) e^{t-\tau_U}\, dt = \Psi_U S_U + \Psi_L S_L + \Psi_D S_D$$

Integrating by parts

$$I^+(\tau_{l-1}) = I^+(\tau_l)e^{-\Delta} + \int_{\tau_{l-1}}^{\tau_l} S(t)e^{-(t-\tau_{l-1})}dt$$

$$I^-(\tau_l) = I^-(\tau_{l-1})e^{-\Delta} + \int_{\tau_{l-1}}^{\tau_l} S(t)e^{-(\tau_l-t)}dt$$

we have the integral $\int_{\tau_U}^{\tau_L} S(t)e^{t-\tau_U}\,dt$ expressed in terms of S and its derivatives in ***two points***:

$$I^+(\tau_{l-1}) = I^+(\tau_l)e^{-\Delta} + [S(\tau_{l-1}) + S'(\tau_{l-1}) + S''(\tau_{l-1}) + ...] -$$

$$[S(\tau_l) + S'(\tau_l) + S''(\tau_l) + ...]e^{-\Delta},$$

$$I^-(\tau_l) = I^-(\tau_{l-1})e^{-\Delta} + [S(\tau_l) - S'(\tau_l) + S''(\tau_l) - ...] -$$

$$[S(\tau_{l-1}) - S'(\tau_{l-1}) + S''(\tau_{l-1}) - ...]e^{-\Delta}.$$

- A piece-wise linear interpolation of $S(\tau)$ between τ_{l-1} and τ_l

$$S'(\tau_{l-1}) = S'(\tau_l) = \frac{S(\tau_l) - S(\tau_{l-1})}{\Delta\tau_l}$$

leads to

$$I^-(\tau_l) = I^-(\tau_{l-1})e^{-\Delta} + p_l \cdot S(\tau_l) + q_l \cdot S(\tau_{l-1})$$

$$I^+(\tau_{l-1}) = I^+(\tau_l)e^{-\Delta} + p_l \cdot S(\tau_{l-1}) + q_l \cdot S(\tau_l)$$

- A piece-wise parabolic interpolation of $S(\tau)$ between τ_{l-1} and τ_l

$$S''(\tau_l) = S''(\tau_{l-1}) = \frac{S'(\tau_l) - S'(\tau_{l-1})}{\Delta\tau_l}$$

leads to

$$I^-(\tau_l) = I^-(\tau_{l-1})e^{-\Delta} + p_l \cdot S(\tau_l) + q_l \cdot S(\tau_{l-1}) + r_l \cdot S'(\tau_l) + s_l \cdot S'(\tau_{l-1})$$

$$I^+(\tau_{l-1}) = I^+(\tau_l)e^{-\Delta} + p_l \cdot S(\tau_{l-1}) + q_l \cdot S(\tau_l) - r_l \cdot S'(\tau_{l-1}) - s_l \cdot S'(\tau_l)$$

Hermite interpolation

matches both the function and the derivative values at the ends of the interval.

Hermite polynomial on the interval (τ_0, τ_1)

$$S(\tau) = a_1 S_0 + a_2 S_1 + \Delta\tau(a_3 S_0' + a_4 S_1')$$

Bezier splines

go through the end points of the interval, but use control points S_c inside the interval to shape the polynomial curve.

Basis functions: $((1-u)+u)^n = (1-u)^n + n(1-u)^{n-1}u + ... + n(1-u)u^{n-1} + u^n$

Quadratic (n=2)
Bezier spline:

$$S(u) = (1-u)^2\, S_0 + u^2\, S_1 + 2u(1-u)\, S_c$$

$$I(\tau_0) = I(\tau_1)e^{-(\tau_1-\tau_0)} + \alpha_0\, S_0 + \beta_0\, S_1 + \gamma_0\, S_d$$

Cubic (n=3)
Bezier spline: $S(u) \approx (1-u)^3\, S_0 + u^3\, S_1 + 3u(1-u)^2\, S_c + 3u^2(1-u)\, S_d$

Integral form of RTE – short characteristics approach

RTE "along a ray"

$$\frac{dI}{d\tau} = I - S$$

The formal solution of this equation (integrating from point *L-1* to *L*)

$$I_L = \underbrace{I_{L-1}e^{-\Delta}}_{\text{part transmitted from point } L\text{-}1 \text{ to point } L} + \int_0^{\Delta} S(t)e^{t-\Delta}dt$$

Polynomial representation of the source function leads either to:

$$I_L = I_{L-1}e^{-\Delta} + \boxed{\omega_{L-1}S_{L-1} + \omega_L S_L + \omega_{L+1}S_{L+1}}$$ (ALI: 3-point algorithm)

or

$$I_L = I_{L-1}e^{-\Delta} + \boxed{pS_L + qS_{L-1} + rS'_L}$$ (FBILI: 2-point algorithm)