Time series analysis

Timing methods
in X-ray Astronomy

Timing features
in X-ray Astronomy

TIME SERIES ANALYSIS

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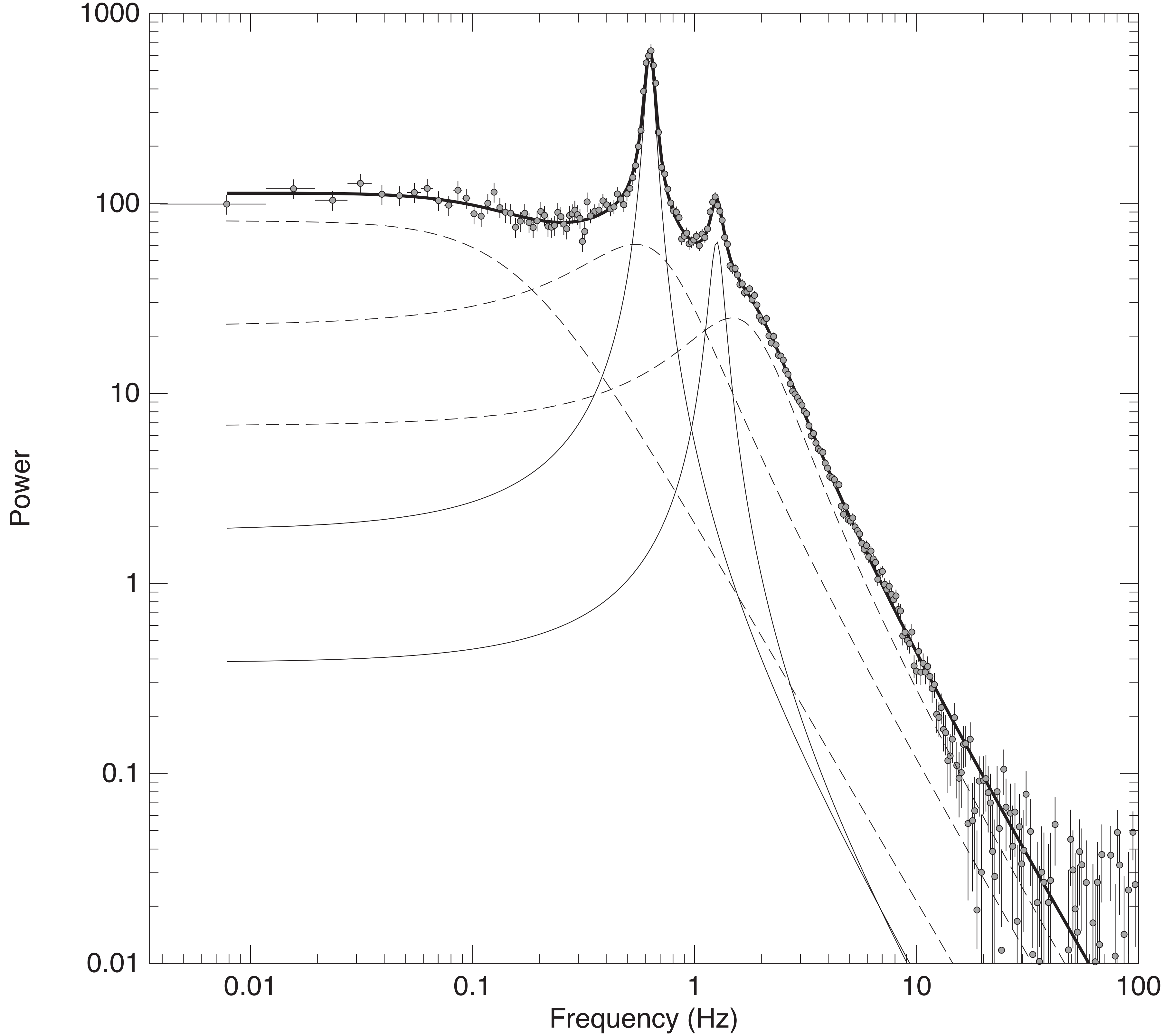
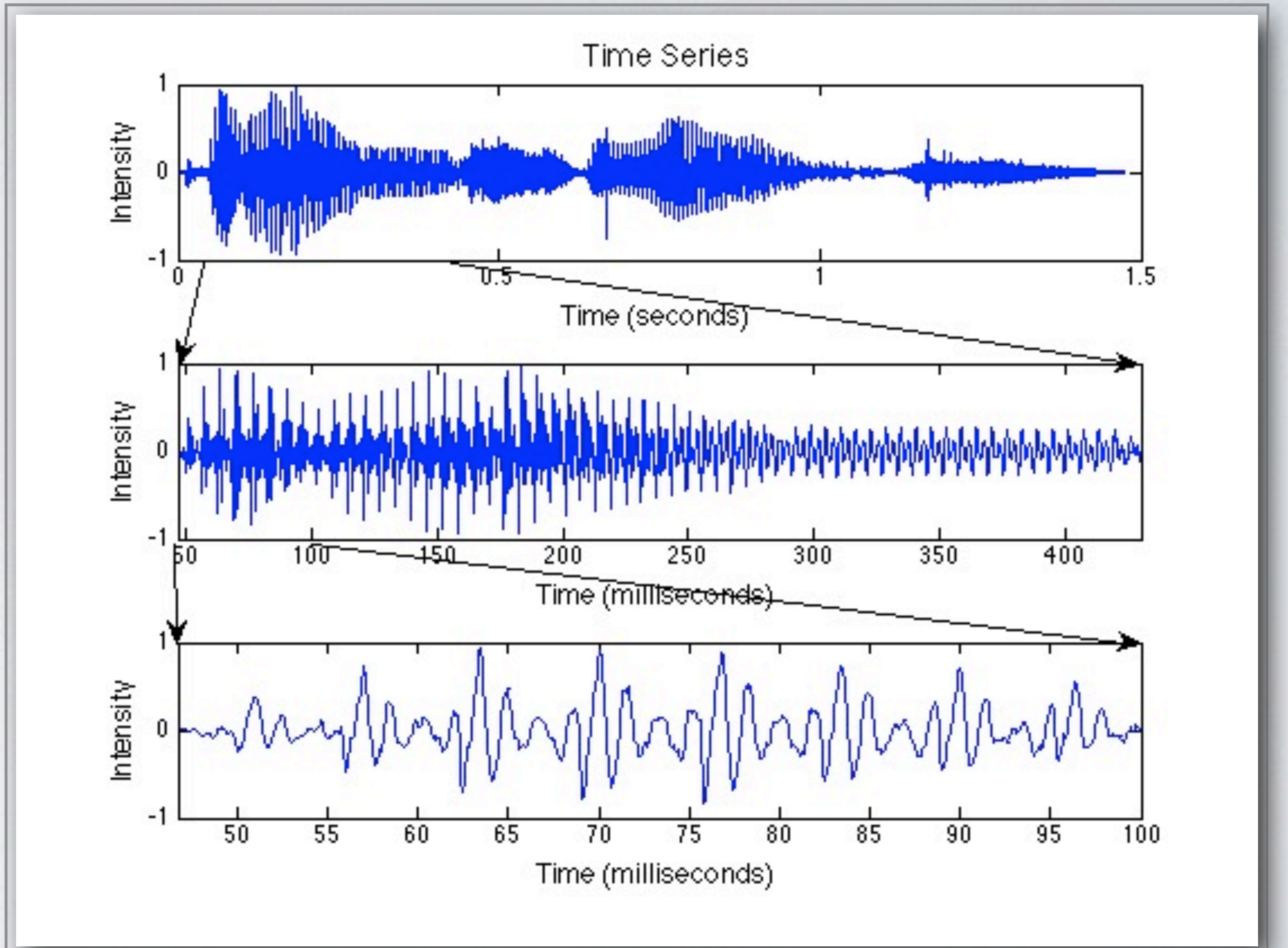


1000
100
10
1
0.1
0.01
Power
0.01
0.1
1
10
100
Frequency (Hz)

TIME SERIES

$$h(t) \quad \text{or} \quad z(t)$$

- 1-d sequence
- Many obvious examples
- Large literature on many fields



TIME SERIES AND FREQUENCY

Joseph Fourier (1768-1830)

- Time is important
- Different representation
- Frequency domain
- Fourier analysis
- Heat conduction → discontinuity

FOURIER TRANSFORM

- Fourier transform equations

- h(t) and H(f): two representations of the same equation

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{2\pi i f t} df$$

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{-2\pi i f t} dt$$

- Linear transformation

- Decomposition on sine waves

- $\sin(2\pi f_0 t) \Leftrightarrow \delta(f-f_0)$

- Invariant to time shift

h(t)	H(f)
Real	H(-f) = [H(f)]*
Even	H(-f) = H(f) [even]
Odd	H(-f) = -H(f) [odd]
Real & Even	H(f) is real and even
Real & Odd	H(f) is imaginary and odd

OTHER BASIC PROPERTIES

- Correlation

$$Corr(g,h) = \int_{-\infty}^{\infty} g(t+\tau)h(\tau)d\tau \iff G(f)H^*(f)$$

- Autocorrelation

Autocorrelation is the fourier transform of the power spectrum

$$Corr(g,g) = \int_{-\infty}^{\infty} g(t+\tau)g(\tau)d\tau \iff |G(f)|^2$$

- Parseval's theorem

$$\int_{-\infty}^{\infty} |h(t)|^2 dt = \int_{-\infty}^{\infty} |H(f)|^2 df$$

Total power in the signal

ONE-SIDED VS. TWO-SIDED

- Power spectral density (PSD)

$$P_h(f) \equiv |H(f)|^2 \qquad -\infty < f < \infty$$

- One-sided

$$P_h(f) \equiv |H(f)|^2 + |H(-f)|^2 \qquad 0 \leq f < \infty$$

- If h(t) is real

$$P_h(f) \equiv 2|H(f)|^2$$

RECAP

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{-2\pi i f t} dt$$

- Fourier transform: decomposition on a base of sinusoids
- Sum of correlation with sinusoids
- h(t) extends from -∞ to +∞
- PSD over frequency gives signal power
- We have real signals...
- ... but we don't have either continuous or infinite signals

DISCRETE FOURIER TRANSFORM

- Sampled function: x_k (k=1,...,N), total length T   [N numbers]

Discrete FT $$a_j = \sum_{k=0}^{N-1} x_k e^{2\pi ijk/N}$$ (j=-N/2,...,N/2-1)

- Here times are t_k = kT/N, frequencies are j/T

- Time step: ΔT = T/N

- Frequency step: Δν = 1/T

Inverse FT

$$x_k = \frac{1}{N} \sum_{j=-N/2}^{N/2-1} a_j e^{-2\pi ijk/N}$$

UNCERTAINTY PRINCIPLE (I)

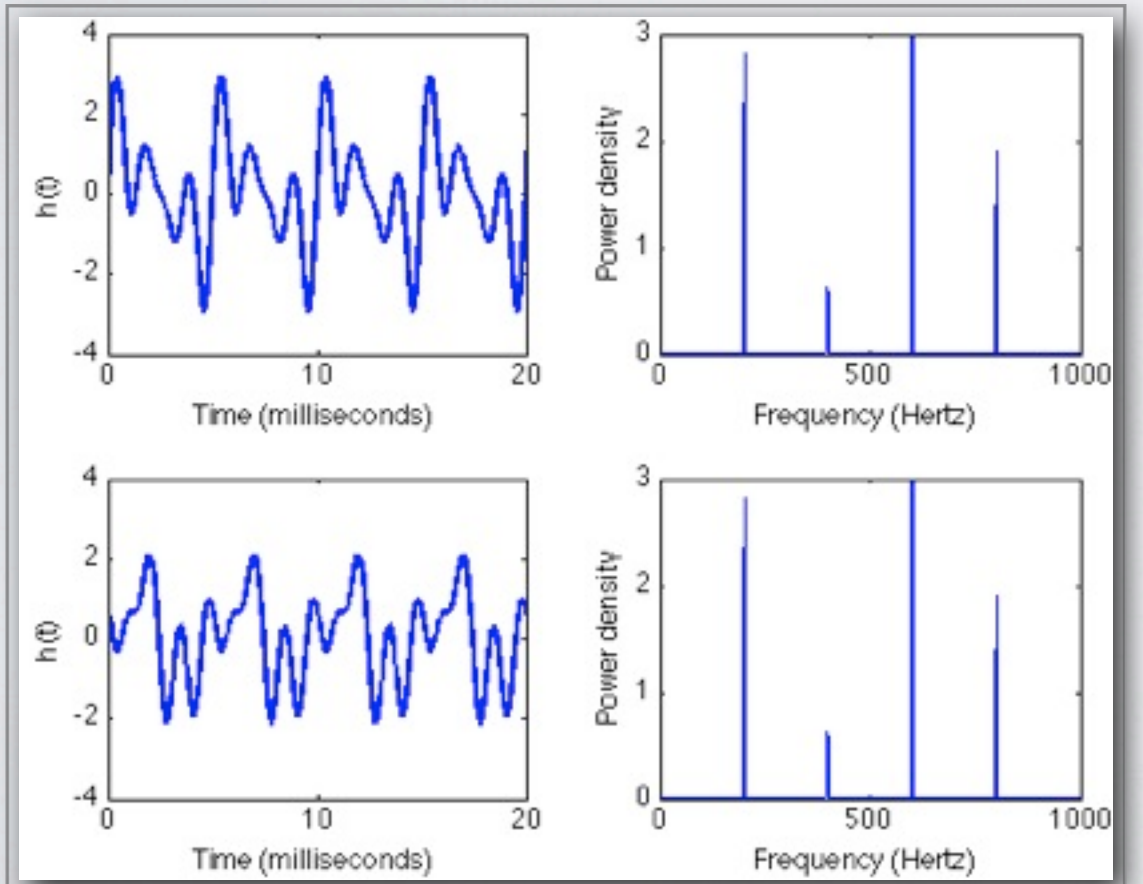
- Frequency resolution: $\Delta\nu = 1/T$
- Time resolution: T (length of the sample of N measurements)
- The longer your measurement, the higher your frequency resolution
- This is important in time-frequency analysis (non-stationary signals)
- Formal version of UP much more complex

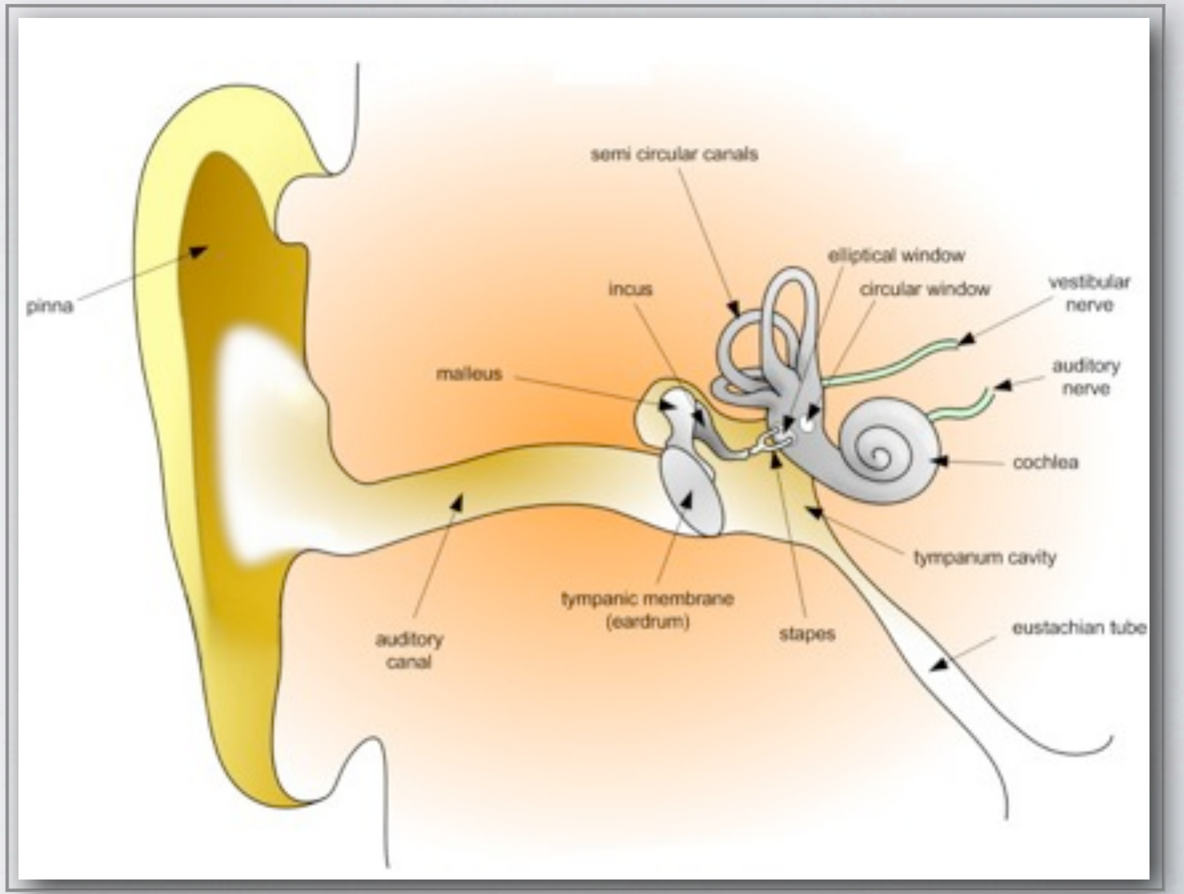
NO LOSS OF INFORMATION

- N numbers in input - N numbers in output (for real signals H(-f) = [H(f)]*, but values are complex)
- Highest frequency: $\nu_{N/2} = \frac{1}{2}\frac{N}{T}$ Nyquist frequency
- Critical sampling of a sine wave is two sample points per cycle
- If you sample less, you get the wrong period (wait..)
- Notice that H(f) is complex for real input
- Also: $a_0 = \sum_{k=0}^{N-1} x_k e^{2\pi i 0 k/N} = \sum_k x_k \equiv N_{counts}$

POWER DENSITY SPECTRUM

- If we ignore the phases of the a_j's: $$P = \frac{2}{N_{phot}}|a|^2$$ (j=0,...,N/2)
- Again, analogous to hearing system



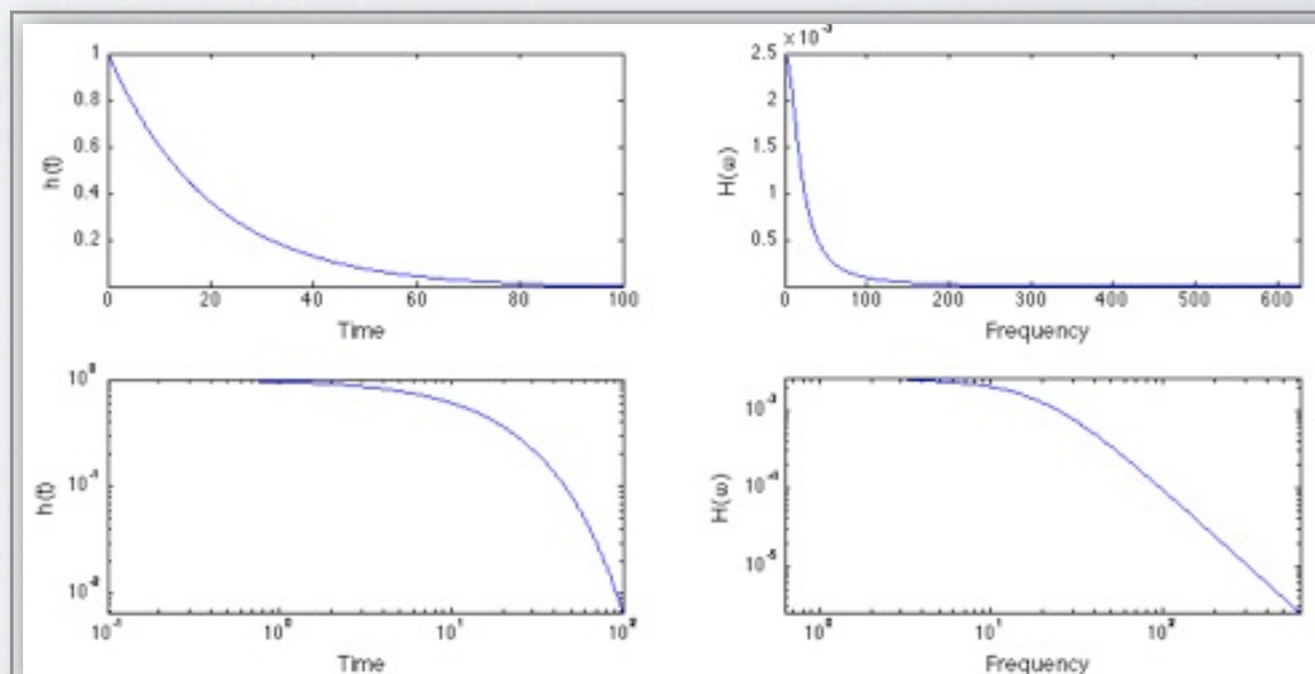


POWER DENSITY SPECTRUM

- An example: (continuous) transform of a one-sided exponential

$$h(t) = e^{-\lambda t} \qquad H(f) = \frac{1}{2\pi i f + \lambda} \equiv \frac{1}{i\omega + \lambda}$$

$$P(f) = |H(f)|^2 = \frac{1}{\omega^2 + \lambda^2}$$



POWER DENSITY SPECTRUM

- Non-linear transformation

$$x_k = y_k + z_k \quad \Rightarrow \quad \begin{aligned} a_j &= b_j + c_j \\ |a_j|^2 &= |b_j|^2 + |c_j|^2 + crossterms \end{aligned}$$

- If independent (random noise added), cross terms average out to zero

FINITE DURATION AND SAMPLING

- How can one connect continuous and discrete FT?

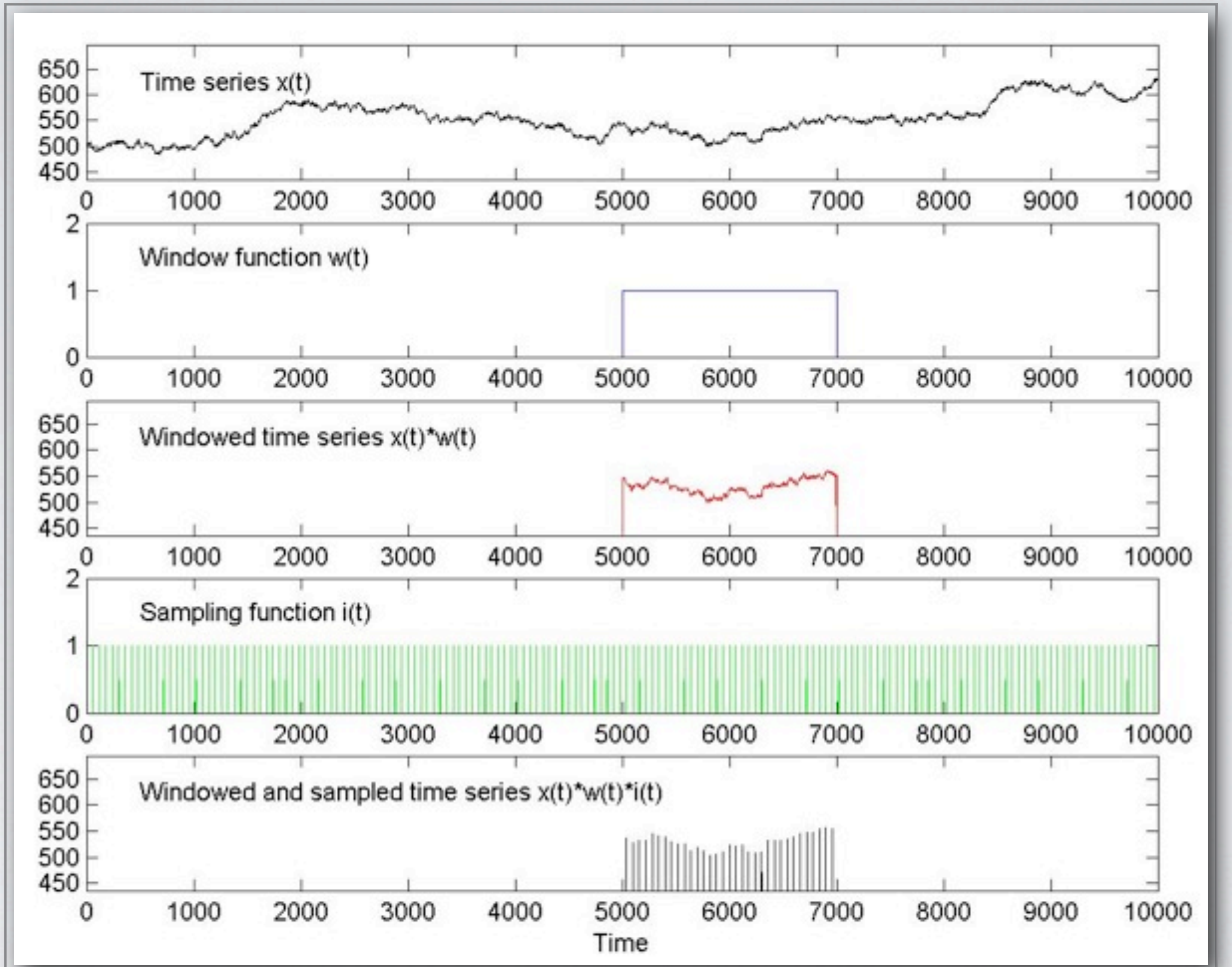
$$a(\nu) = \int_{-\infty}^{\infty} h(t) e^{-2\pi i \nu t} dt \qquad a_j = \sum_{k=0}^{N-1} h_k e^{-2\pi i j k / N}$$

- Continuous time series: *h(t) [-∞,+∞]*
- Discrete time series: h_k *[k=0, ... , N-1]*

FINITE DURATION AND SAMPLING

- We multiply $$h_k = h(t)w(t)i(t)$$
- *w(t)*: window function $$w(t) = \begin{cases} 1 & 0 \leq t < T \\ 0 & otherwise \end{cases}$$
- *i(t)*: sampling function $$i(t) = \sum_{k=-\infty}^{\infty} \delta(t - \frac{kT}{N})$$

FINITE DURATION AND SAMPLING



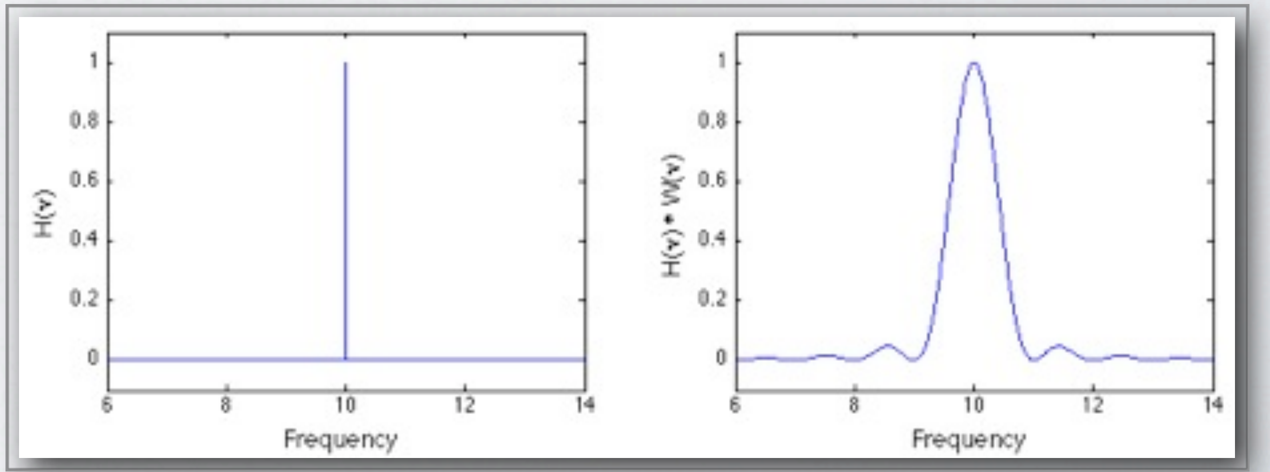
CONVOLUTION THEOREM: WINDOWS

- The transform of the product of two functions is the convolution of the transforms

$$x(t)y(t) \Longleftrightarrow a(\nu) * b(\nu) \equiv \int_{-\infty}^{\infty} a(\mu)b(\nu - \mu)d\mu$$

$$|W(\nu)|^2 \equiv \left| \int_{-\infty}^{\infty} w(t)e^{-2\pi\nu T} \right|^2 = \left| \frac{\sin \pi\nu T}{\pi\nu} \right|^2$$

Broadening of peaks

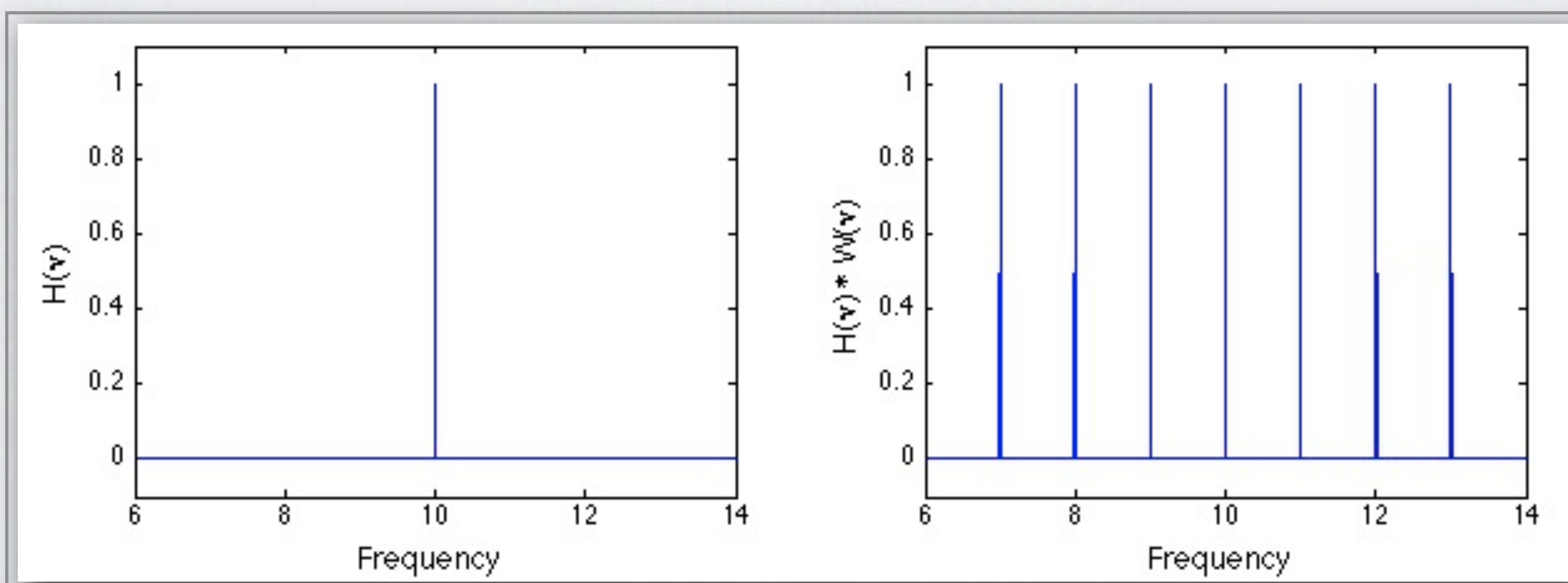


CONVOLUTION THEOREM: SAMPLING

- The transform of the product of two functions is the convolution of the transforms

$$I(\nu) \equiv \int_{-\infty}^{\infty} i(t) e^{-2\pi\nu i t} dt = \frac{N}{T} \sum_{\ell=-\infty}^{\infty} \delta\left(\nu - \ell \frac{N}{T}\right)$$

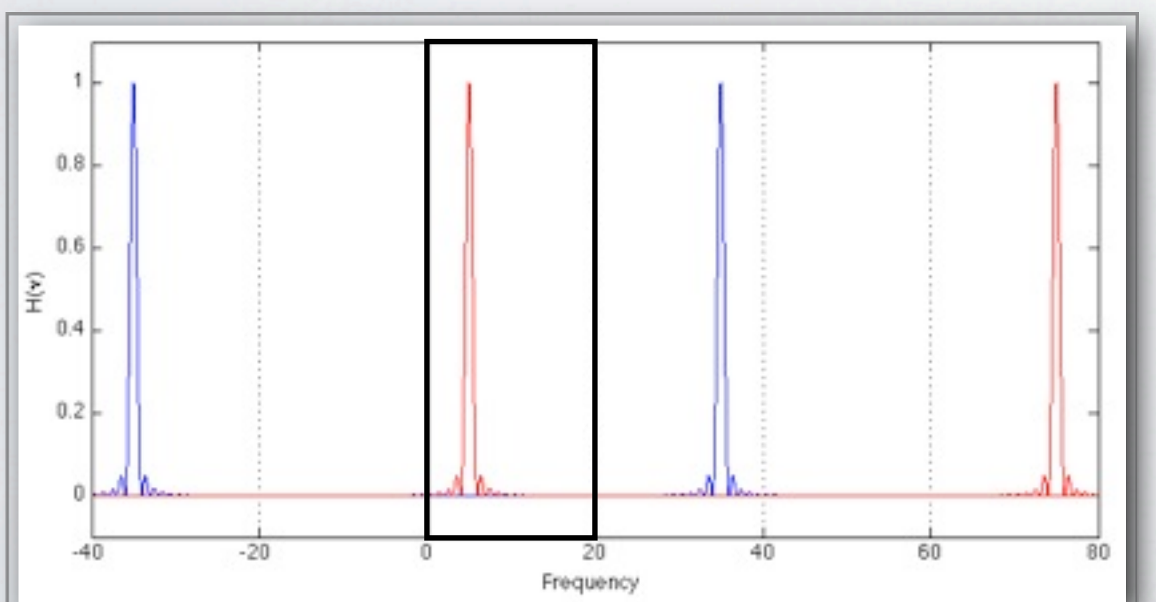
- Infinite series of **δ** functions, with spacing $N/T = 2\,\nu_{Nyq}$

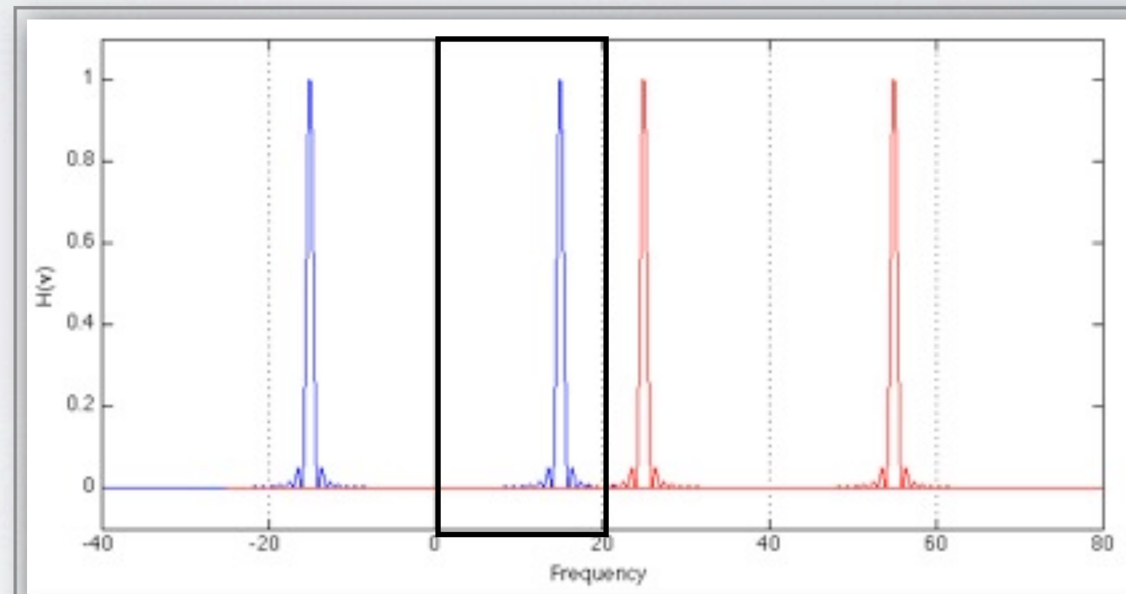


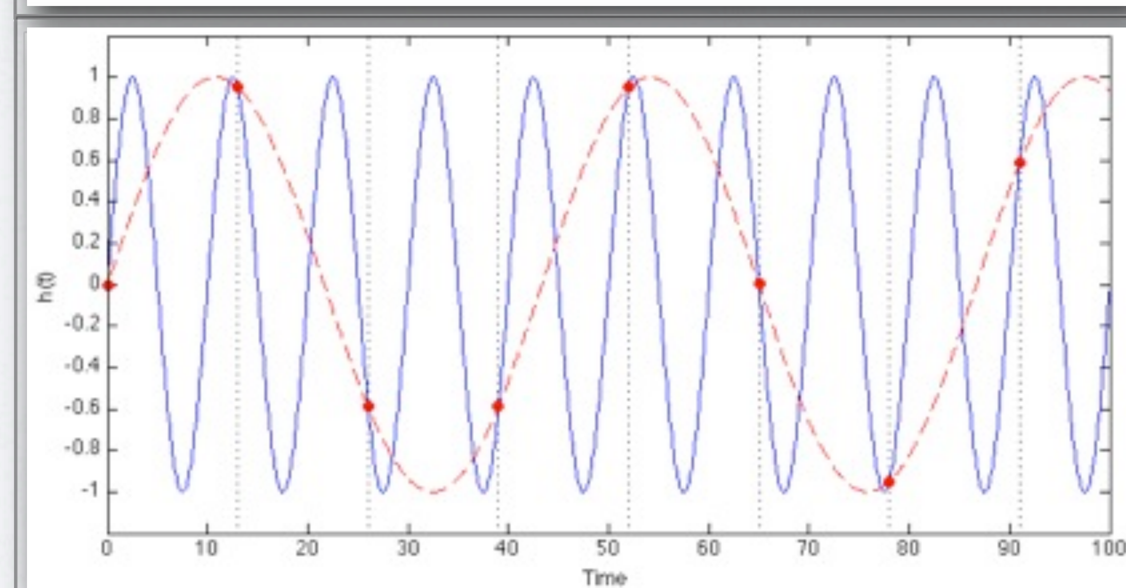
$>\nu_{Nyq}!$

ALIASING

- FT is symmetric in frequency for a real signal
- Alias repeats it every $2\boldsymbol{\nu}_{Nyq}$
- Problem is signal above $\boldsymbol{\nu}_{Nyq}$







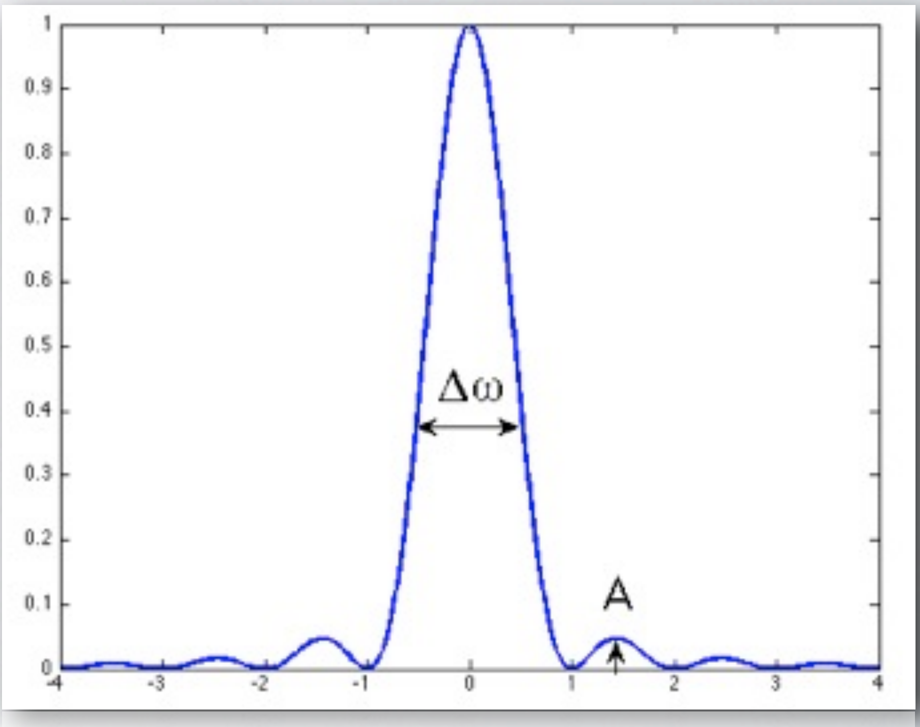
SUMMARY OF DISCRETE FT EFFECTS

- WINDOW: broadening & sidebands
- SAMPLING: aliasing

- Aliasing not such a big problem for high-energy astronomy
- Binning, not sampling
- Suppression of high frequencies

WINDOW EFFECTS

- Window effect is a problem:
  - It broadens delta peaks
  - It flattens the slopes of noise components (sidelobes)
- The longer the observation, the better

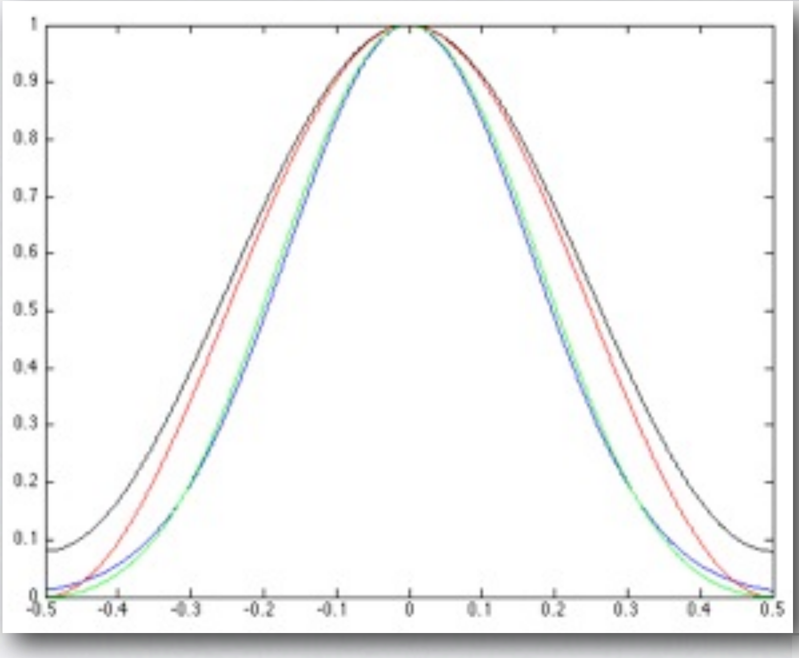


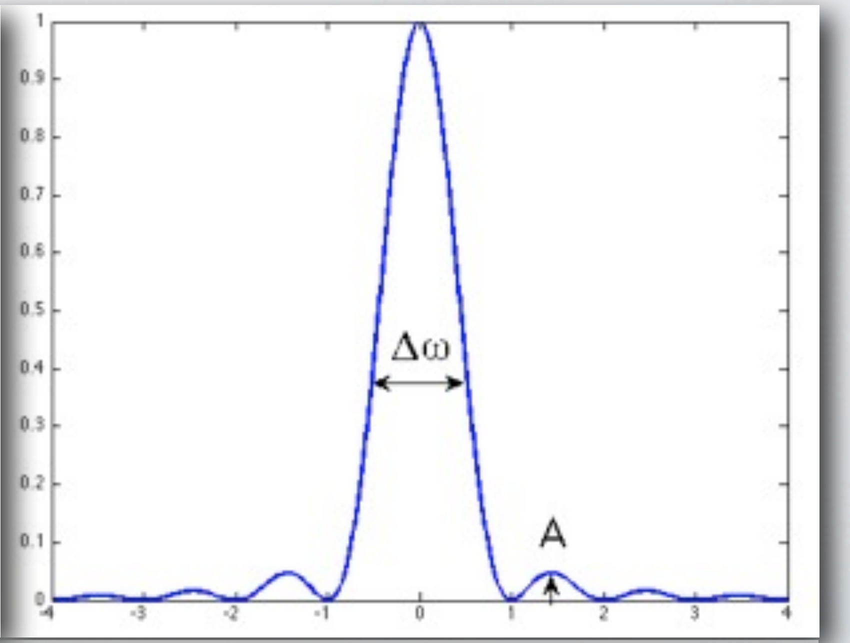
WINDOW CARPENTRY

- ✤We can use different windows

Window	$\Delta\omega$	A	p	Function
Boxcar	0.89	-13db	2	1
Hamming	1.36	-43db	2	$0.54+0.46\cos(2\pi t)$
Gaussian	1.55	-55db	2	$\exp(-18t^2)$
Hanning	1.44	-32db	5	$\cos^2(\pi t)$
Blackman	1.68	-58db	5	$0.42+0.5\cos(2\pi t)+0.08\cos(4\pi t)$

- ✤We lose some signal



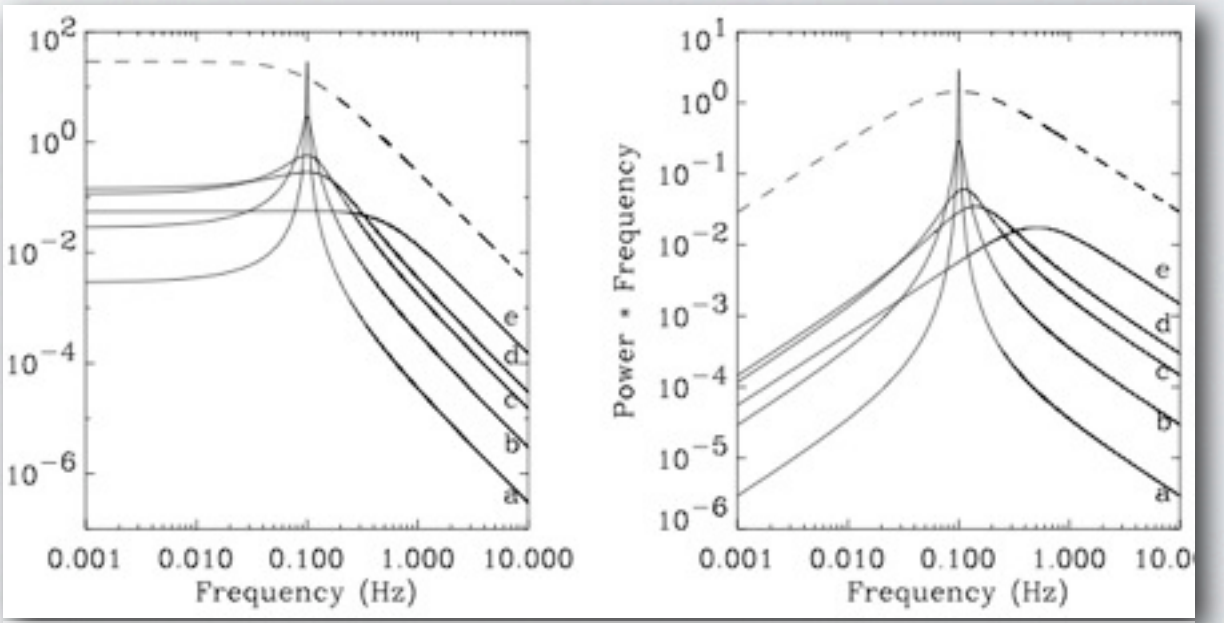


POWER SPECTRA UNITS

- A power spectrum is in units of Hz^{-1}
- It scales with the square of the intensity: variance
- If we divide by the square of the intensity, we get the fractional variance (squared rms)
- The square root of its integral is the total fractional rms
- Useful to compare amount of variability

POWER SPECTRUM PLOTS

- Multiply the power spectrum by the frequency
- Obtain a νP_ν representation
- Useful to see where the power per decade peaks
- Characteristic frequencies are peaks in νP_ν (later)



FAST FOURIER TRANSFORM

- What is the Fourier Transform of a single point?
- Split the series in two: odd and even points
- The FT of the series can be expressed (simply) from the FT of the two subseries
- Repeat
- Reach 1
- Reassemble

$O(N^2)$ $O(N\log_2 N)$

FURTHER READING

- http://web.me.com/tbelloni/Timing/Home.html
- M. van der Klis: Fourier Techniques in X-Ray Timing (technical)
- Tilman Butz: Fourier Transformations for Pedestrians (Springer)
- Numerical Recipes (as usual)