Second part of the talk

Homogeneous vs Inhomogeneous

- No matter where they come from, PMFs could have left an imprint on the CMB anisotropy pattern in temperature and polarization.
- In case of an homogeneous PMF, our Universe would be described by the Bianchi metric. Barrow, Ferreira & Silk 1997
- Inhomogenous PMFs (respecting homogeneity and isotropy globally) can source scalar, vector and tensor non-gaussian perturbations.
- CMB predictions in presence of a SB of PMFs has drawn a lot of interest as a non-standard cosmological model beyond the simplest ΛCDM rich of observational signatures (B-mode polarization from the vector contribution, Faraday rotation, non-gaussianities, effects due to non-zero helicity): most of these effects are larger at high multipoles, which have been characterized better and better by several experiments in the last years (ACBAR, SPT, ACT, Planck ....).
- Affecting cosmological perturbations, PMF could leave an imprint also on the LSS.

Inhomogeneous Magnetic Field

- A SB of PMFs is modeled as a fully inhomogeneous component (i.e. does not affect the evolution of the Hubble parameter).

- The infinite conductivity limit is assumed (E is zero and the SB is comoving with the expansion of the universe).
- On considering B^2 at the same level of metric and density fluctuations, PMFs affects matter and metric perturbations in three ways:

a. PMFs carry energy density and pressure and therefore gravitate at the level of perturbations.

b. PMFs have anisotropic stress - differently from perfect fluids - which adds to the photon and neutrino ones (the photon anisotropic stress is negligible before the decoupling epoch). This anisotropic stress has scalar, vector and tensor component.

c. PMFs induce a Lorentz force on baryons, which indirectly affects also photons during the tight coupling regime. The Lorentz force has a scalar and vector component.

- PMFs therefore lead to inhomogeneous solutions to scalar, vector and tensor metric and matter perturbations.

- These inhomogeneous solutions inherit the statistics of the source terms in B^2 which is non-Gaussian (more precisely a chi-square distribution) and this is an important point for the imprint on CMB non-gaussianities.

The relevant GR equations

- Einstein equations: $$G_{\mu\nu} = 8\pi G(T_{\mu\nu} + T_{\mu\nu}^{\rm PMF})$$

- Due to the infinite conductivity limit

  a. the induced electric field vanishes

  b. the magnetic field is frozen in: $$\mathbf{B}(\mathbf{x},\tau) = \frac{\mathbf{B}(\mathbf{x})}{a^2(\tau)}$$

$$T_0^{0\,\rm PMF} = -\rho_B = -\frac{|\mathbf{B}(\mathbf{x})|^2}{8\pi a^4}$$

$$T_i^{0\,\rm PMF} = 0$$

$$T_j^{i\,\rm PMF} = \frac{1}{4\pi a^4}\left(\frac{|\mathbf{B}(\mathbf{x})|^2}{2}\delta_j^i - B_j(\mathbf{x})B^i(\mathbf{x})\right)$$

$$= \delta_j^i \frac{\rho_B}{3} + \Sigma_j^i \qquad \Sigma_i^i = 0$$

- As already stated, $T_{\mu\nu}^{\rm PMF}$ is a stiff source of the Einstein equations acting only at the level of perturbations (same treatment of a network of topological defects, a similar component which also does not respect the symmetries of isotropy and homogeneity locally).

- The presence of PMFs in a globally neutral plasma induces a Lorentz force on baryons:

$$\mathbf{L}(\mathbf{x},\tau) = \frac{\mathbf{L}(\mathbf{x})}{a^4} \qquad L_i(\mathbf{x}) = \frac{1}{4\pi}\left[B_j(\mathbf{x})\nabla_j B_i(\mathbf{x}) - \frac{1}{2}\nabla_i(B_j(\mathbf{x})B^j(\mathbf{x}))\right]$$

Metric Perturbations

- Inhomogeneous perturbations around homogeneous background:

$$g_{\mu\nu}(\mathbf{x},\tau) = g^{(0)}_{\mu\nu}(\tau) + \delta g_{\mu\nu}(\mathbf{x},\tau)$$
$$T_{\mu\nu}(\mathbf{x},\tau) = T^{(0)}_{\mu\nu}(\tau) + \delta T_{\mu\nu}(\mathbf{x},\tau)$$

For simplicity flat spatial sections: $g^{(0)}_{\mu\nu}(\tau) = a^2(\tau)\,\mathrm{diag}\,(-,+,+,+)$

All except PMF have homogeneous EMT: non relativistic components (baryons, CDM), relativistic components (radiation, massless neutrinos), dark energy (only homogeneous if Λ).

- Metric perturbations have ten degrees of freedom (dofs) which can be divided in scalar, vector and tensor as for any symmetric tensor:

$$A_{ij} = A\delta_{ij} + \left(\partial_i\partial_j - \frac{\delta_{ij}}{3}\nabla^2\right)\bar{A} + A^V_{ij} + A^T_{ij}$$

Trace

Scalar potential for the traceless part

Vector part

$$A^V_{ij} = \partial_i A^V_j + \partial_j A^V_i$$

Tensor part: traceless and transverse

$$A^{T\,i}_i = \partial_i A^{T\,i}_j = 0$$

A^V_i Vector potential $\partial_i A^{V\,i} = 0$

- The ten metric perturbations (4 scalar, 4 vector and 2 tensor) are redundant and is convenient to keep separate the physical perturbations from the infinitesimal changes in the coordinate system (called **gauge**) transformations:

$$\tilde{x}^{\mu} = x^{\mu} + \xi^{\mu}(x)$$

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - \nabla_{\mu}\xi_{\nu} - \nabla_{\nu}\xi_{\mu}$$

- The four dofs of the infinitesimal coordinate transformation corresponds to 2 scalar and 2 vector functions. The tensor part is not affected by gauge transformations and is directly **gauge** invariant.

- This split into scalar, vector and tensor components holds as well for the PMF anisotropic stress $\sum^{i}_{j}$ (with the difference that is traceless).

A stochastic background of PMF

- We assume the magnetic field satisfy in Fourier space:

$$\langle B_i(\mathbf{k}) B_j^*(\mathbf{k}')\rangle = \frac{(2\pi)^3}{2}\delta(\mathbf{k}-\mathbf{k}')\Big[(\delta_{ij}-\hat{k}_i\hat{k}_j)P_S(k) + i\epsilon_{ijl}\hat{k}_l P_A(k)\Big]$$

- The power spectra are related to the symmetric part and to the helical part:

$$\langle B_i(\mathbf{k}) B_i^*(\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')P_S(k)$$
$$\langle (\nabla\times\mathbf{B})_i(\mathbf{k}) B_i^*(\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')P_A(k)$$

- Above definition of helicity is gauge-invariant, similar to kinetic helicity.

- Other definitions of helicity are available as

$$\langle A_i(\mathbf{k}) B_i^*(\mathbf{k}')\rangle \qquad \text{with} \quad \mathbf{B} = \nabla\times\mathbf{A}$$

which is not gauge invariant unless the normal direction of the B field perpendicular to the surface decays sufficiently faster. With this latter definition, the helicity would be constant due to the large conductivity.

- Power spectrum smoothed on a scale λ (tipically chosen as 1 Mpc).

$$B_\lambda^2 = \int \frac{d^3k}{(2\pi)^3} P_S(k) \exp(-\lambda^2 k^2) = \frac{A_S}{(2\pi)^2} \frac{1}{\lambda^{n_S+3}} \Gamma\Big(\frac{n_S+3}{2}\Big)$$

$$\mathcal{B}_\lambda^2 = \lambda \int \frac{d^3k}{(2\pi)^3} k |P_A(k)| \exp(-\lambda^2 k^2) = \frac{|A_A|}{(2\pi)^2} \frac{1}{\lambda^{n_A+3}} \Gamma\Big(\frac{n_A+4}{2}\Big)$$

- Also used the mean square of B:

$$\langle B^2 \rangle = \int \frac{d^3k}{(2\pi)^3} P_S(k) = \frac{A_S}{2\pi^2 (n_S+3)} \frac{k_D^{n_S+3}}{k_*^{n_S}} = B_\lambda^2 \frac{k_D^{n_S+3} \lambda^{n_S+3}}{(n_S+3)\Gamma\Big(\frac{n_S+3}{2}\Big)}$$

- The helical term must satisfies a Schwarz inequality:

$$|P_A(k)| \leq P_S(k)$$

- Example of a coupling to a pseudo-scalar field in which the helicity basis is convenient:

$$P_A \propto |A_+|^2 - |A_-|^2$$

$$P_S \propto |A_+|^2 + |A_-|^2$$

- Analysis including helicity in preparation, selected comparison with non-helical results for completeness.

Power-spectra of the PMF EMT

- We assume the magnetic field satisfy (no helicity is considered, note the change in notation):

$$\langle B_i(\mathbf{k}) B_j^*(\mathbf{k}')\rangle = (2\pi)^3 \delta(\mathbf{k}-\mathbf{k}')(\delta_{ij} - \hat{k}_i \hat{k}_j)\frac{P_B(k)}{2}$$

- Power spectrum damped at small scales larger than k_D by radiation viscosity:

$$P_B(k) = A\Big(\frac{k}{k_*}\Big)^{n_B} \neq 0 \quad \text{only for } k < k_D \,,$$

where k_* is a pivot scale and k_D is a damping scale.

- The spectra of the energy momentum components of the PMF are 4-point functions of the magnetic field.

- Example of energy density:

$$\langle \rho_B(\mathbf{k}) \rho_B^*(\mathbf{k}')\rangle = (2\pi)^3 \delta(\mathbf{k}-\mathbf{k}')|\rho_B(k)|^2$$

$$\langle \rho_B(\mathbf{k}) \rho_B^*(\mathbf{k}')\rangle \propto \int d\mathbf{p} \int d\mathbf{q} \langle B_i(\mathbf{k}) B_i(\mathbf{k}-\mathbf{p}) B_j(\mathbf{k}') B_j(\mathbf{k}'-\mathbf{q})\rangle$$

$$= \int d\mathbf{p} \int d\mathbf{q} \langle B_i(\mathbf{k}) B_j(\mathbf{k}')\rangle \langle B_i(\mathbf{k}-\mathbf{p}) B_j(\mathbf{k}'-\mathbf{q})\rangle +$$

$$\langle B_i(\mathbf{k}) B_j(\mathbf{k}'-\mathbf{q})\rangle \langle B_i(\mathbf{k}-\mathbf{p}) B_j(\mathbf{q})\rangle$$

Wick theorem

- Wick theorem (example for four point correlation function):

$$\langle X_1 X_2 X_3 X_4 \rangle = \langle X_1 X_2 \rangle \langle X_3 X_4 \rangle + \langle X_1 X_3 \rangle \langle X_2 X_4 \rangle + \langle X_1 X_3 \rangle \langle X_2 X_4 \rangle$$

- Spatial components of the EMT:

$$\begin{aligned} \langle \tau^*_{ab}(\mathbf{k}) \tau_{cd}(\mathbf{k}') \rangle = & \int \frac{d\mathbf{q} d\mathbf{p}}{64\pi^5} \delta_{ab}\delta_{cd} \langle B_l(\mathbf{q}) B_l(\mathbf{k}-\mathbf{q}) B_m(\mathbf{p}) B_m(\mathbf{k}'-\mathbf{p}) \rangle \\ & - \int \frac{d\mathbf{q} d\mathbf{p}}{32\pi^5} \langle B_a(\mathbf{q}) B_b(\mathbf{k}-\mathbf{q}) B_c(\mathbf{p}) B_d(\mathbf{k}'-\mathbf{p}) \rangle \end{aligned}$$

- Scalar, vector and tensor of the spatial anisotropic stress:

$$\begin{aligned} \langle \Pi^{*(S)}(\mathbf{k}) \Pi^{(S)}(\mathbf{k}') \rangle &= (2\pi)^3 |\Pi^{(S)}(k)|^2 \delta(\mathbf{k}-\mathbf{k}') = \delta_{ab}\delta_{cd} \langle \tau^*_{ab}(\mathbf{k}) \tau_{cd}(\mathbf{k}') \rangle \\ \langle \Pi^{*(V)}_i(\mathbf{k}) \Pi^{(V)}_j(\mathbf{k}') \rangle &= \frac{(2\pi)^3}{2} |\Pi^{(V)}(k)|^2 P_{ij}(\mathbf{k}) \delta(\mathbf{k}-\mathbf{k}') = k_a P_{ib}(\mathbf{k}) k'_c P_{jd}(\mathbf{k}') \langle \tau^*_{ab}(\mathbf{k}) \tau_{cd}(\mathbf{k}') \\ \langle \Pi^{*(T)}_{ij}(\mathbf{k}) \Pi^{(T)}_{tl}(\mathbf{k}') \rangle &= \frac{(2\pi)^3}{4} |\Pi^{(T)}(k)|^2 \mathcal{M}_{ijtl}(\mathbf{k}) \delta(\mathbf{k}-\mathbf{k}') \\ &= \left[P_{ia}(\mathbf{k}) P_{jb}(\mathbf{k}) - \frac{1}{2} P_{ij}(\mathbf{k}) P_{ab}(\mathbf{k}) \right] \left[P_{tc}(\mathbf{k}') P_{ld}(\mathbf{k}') - \frac{1}{2} P_{tl}(\mathbf{k}') P_{cd}(\mathbf{k}') \right] \langle \tau^*_{ab}(\mathbf{k}) \tau_{cd}(\mathbf{k}') \rangle \end{aligned}$$

where $\mathcal{M}_{ijtl} = P_{it}P_{jl} + P_{il}P_{jt} - P_{ij}P_{tl}$

- With these inputs:

$$|\rho_B(k)|^2 = \frac{1}{1024\pi^5}\int_\Omega d\mathbf{p} P_B(p)\, P_B(|\mathbf{k}-\mathbf{p}|)(1+\mu^2)$$

$$|L(k)|^2 = \frac{1}{1024\pi^5}\int_\Omega d^3\mathbf{p} P_B(p)\, P_B(|\mathbf{k}-\mathbf{p}|)[1+\mu^2+4\gamma\beta(\gamma\beta-\mu)]$$

$$\langle \rho_B(\mathbf{k}) L_B(\mathbf{k}')\rangle = \frac{\delta(\mathbf{k}-\mathbf{k}')}{1024\pi^5}\int_\Omega d\mathbf{p} P_B(p)\, P_B(|\mathbf{k}-\mathbf{p}|)(1-1(\gamma^2+\beta^2)+2\gamma\beta\mu-\mu^2)$$

$$|\Pi^{(V)}(k)|^2 = \frac{1}{512\pi^5}\int d\mathbf{p} P_B(p)\, P_B(|\mathbf{k}-\mathbf{p}|)[(1+\beta^2)(1-\gamma^2)+\gamma\beta(\mu-\gamma\beta)]$$

$$|\Pi^{(T)}(k)|^2 = \frac{1}{512\pi^5}\int d\mathbf{p} P_B(p)\, P_B(|\mathbf{k}-\mathbf{p}|)(1+2\gamma^2+\gamma^2\beta^2)\,,$$

where

$$\mu = \frac{\mathbf{p}\cdot(\mathbf{k}-\mathbf{p})}{p|\mathbf{k}-\mathbf{p}|}\,, \quad \gamma = \frac{\mathbf{k}\cdot\mathbf{p}}{k\,p}\,, \quad \beta = \frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{p})}{k|\mathbf{k}-\mathbf{p}|}$$

Durrer, Ferreira & Kahniashvili 2000
Mack, Kahniashvili & Kosowsky 2002
Caprini, Durrer & Kahniashvili 2004
FF, Paci & Paoletti 2008

Integration scheme

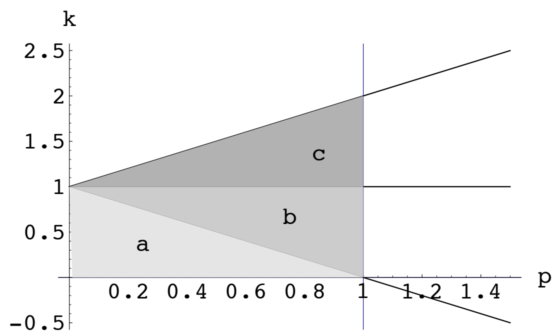
The region of integration is defined where P(k) is non-zero as:

$$p < k_D \qquad\qquad |\mathbf{k} - \mathbf{p}| < k_D$$

The second condition introduces a k-dependence on the angular integration domain and the two allow the energy power spectrum to be non zero only for $0 < k < 2k_D$. Such conditions split the double integral (over γ and over p) in three parts depending on the γ and p lower and upper limit of integration. A sketch of the integration is thus the following:

$$1) \quad 0 < k < k_D : \quad \int_0^{k_D-k} dp \int_{-1}^{1} d\gamma \cdots + \int_{k_D-k}^{k_D} dp \int_{\frac{k^2+p^2-k_D^2}{2kp}}^{1} d\gamma \cdots \equiv \int_0^{k_D-k} dp I_a(p,k) + \int_{k_D-k}^{k_D} dp I_b(p,k)$$

$$2) \quad k_D < k < 2k_D : \int_{k-k_D}^{k_D} dp \int_{\frac{k^2+p^2-k_D^2}{2kp}}^{1} d\gamma \cdots \equiv \int_{k-k_D}^{k_D} dp I_c(p,k)$$

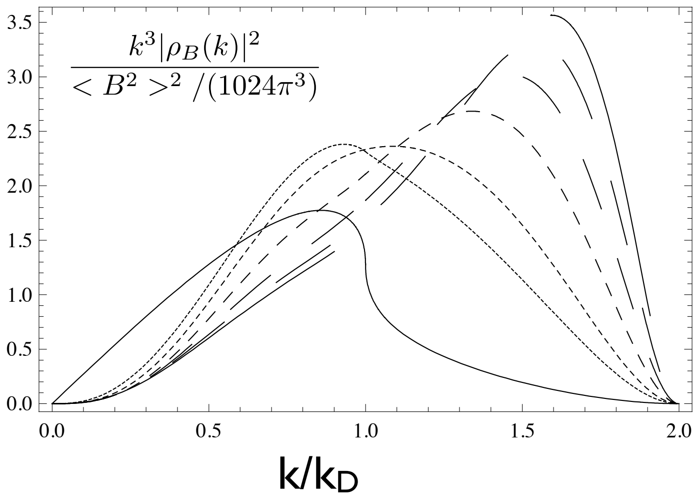


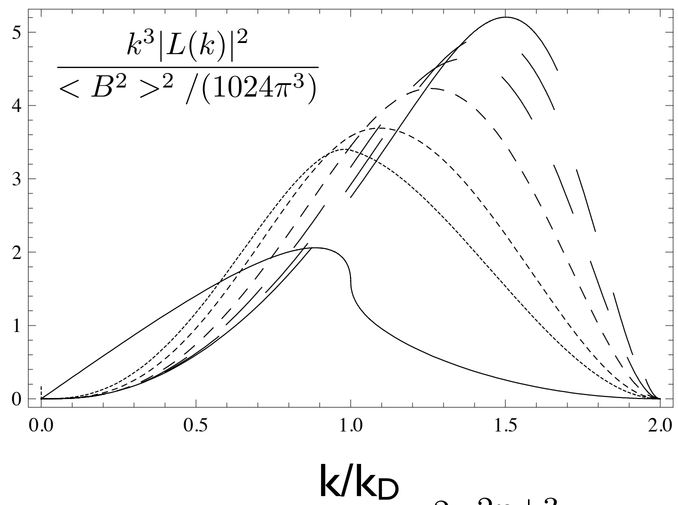
FF, Paci & Paoletti 2008

Careful computation of angular integrals

$$|\rho_B(k)|^2_{n_B=2} = \frac{A^2 k_D^7}{512\pi^4 k_*^4}\left[\frac{4}{7} - \tilde{k} + \frac{8}{15}\tilde{k}^2 - \frac{\tilde{k}^5}{24} + \frac{11}{2240}\tilde{k}^7\right] \qquad \tilde{k} = \frac{k}{k_D}$$

$$|L(k)|^2_{n_B=2} = \frac{A^2 k_D^7}{512\pi^4 k_*^4}\left[\frac{44}{105} - \frac{2\tilde{k}}{3} + \frac{8\tilde{k}^2}{15} - \frac{\tilde{k}^3}{6} - \frac{\tilde{k}^5}{240} + \frac{13\tilde{k}^7}{6720}\right]$$





FF, Paci & Paoletti 2008

n_B=-5/2,-3/2,-1,0,1,2,3

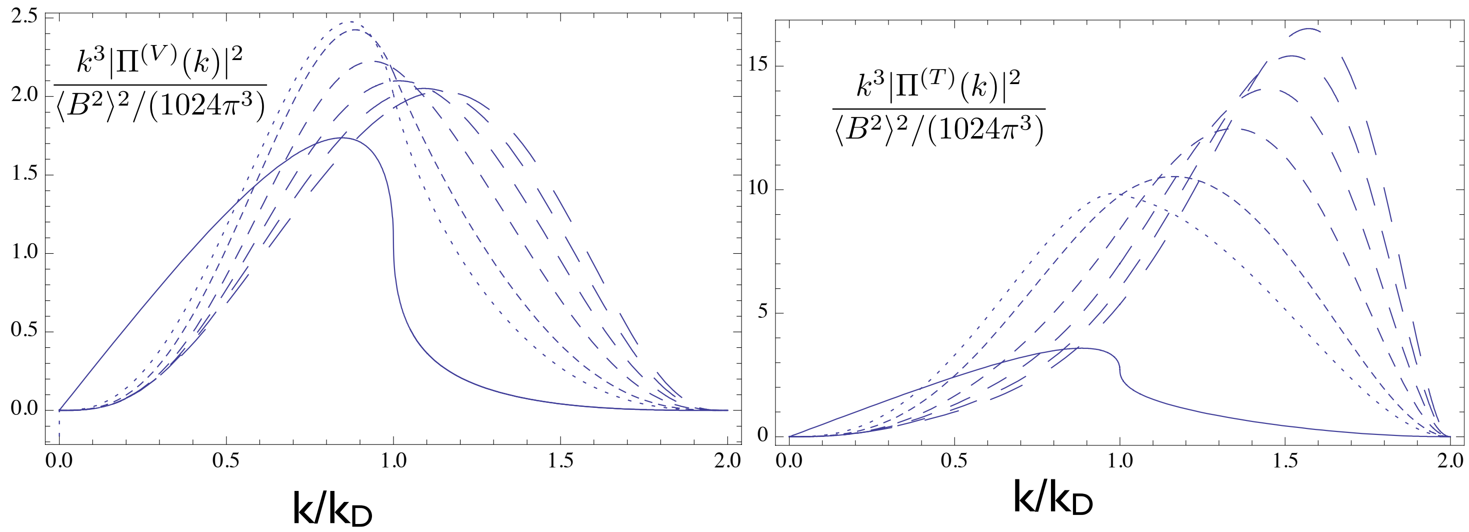
from the solid to large dashed

- For k << k_D and n_B>-3/2 we obtain $|\rho_B(k)|^2 \simeq \frac{A^2 k_D^{2n+3}}{128\pi^4 k_*^{2n}(3+2n_B)}$
- For n_B=-3/2 we have a logarithmic dependence and for -3 < n_B < -3/2 $|\rho_B(k)|^2 \propto k^{2n+3}$
- Differences with respect to previous results in the literature in the infrared part due to the exact computation of the angular part Kahniashvili & Ratra 2007
- There is non-zero correlation between density and Lorentz

$$|\Pi_B^{(V)}(k)|^2_{n_B=2} = \frac{A^2 k_D^7}{256\pi^4 k_*^4}\left[\frac{4}{15} - \frac{5\tilde{k}}{12} + \frac{4\tilde{k}^2}{15} - \frac{\tilde{k}^3}{12} + \frac{7\tilde{k}^5}{960} - \frac{\tilde{k}^7}{1920}\right]$$

$$|\Pi_B^{(T)}(k)|^2_{n_B=2} = \frac{A^2 k_D^7}{256\pi^4 k_*^4}\left[\frac{8}{15} - \frac{7\tilde{k}}{6} + \frac{16\tilde{k}^2}{15} - \frac{7\tilde{k}^3}{24} - \frac{13\tilde{k}^5}{480} + \frac{11\tilde{k}^7}{1920}\right]$$

Similarly we obtain for the vector and tensor part of the anisotropic stress

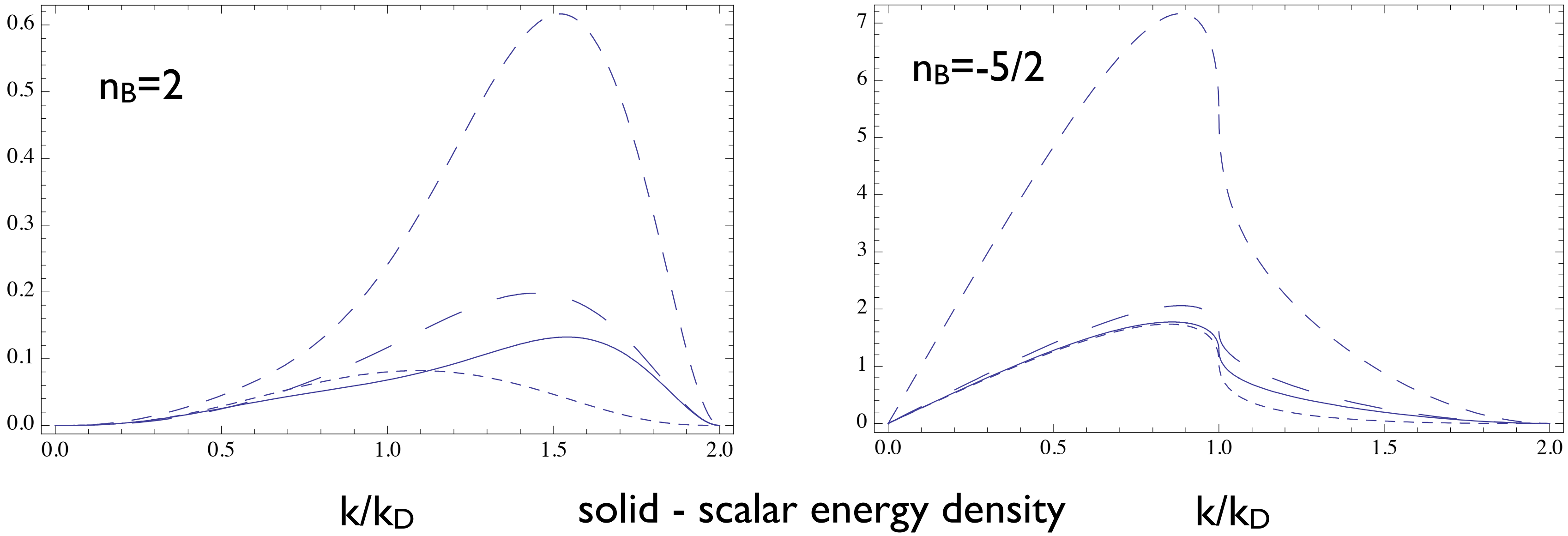


Paoletti, FF & Paci 2009

n_B=-5/2,-3/2,-1,0,1,2,3 from the solid to large dashed

- Differences with respect to previous results in the literature in the infrared part due to the exact computation of the angular part

Mack, Kahniashvili & Kosowsky 2002
Caprini, Durrer & Kahniashvili 2004



solid - scalar energy density
large dashed - scalar Lorentz
short dashed - vector stress
medium dashed - tensor stress

- All contributions (scalar energy density and Lorentz force, vector and tensor anisotropic stress) are comparable

- The various contributions differ because of the transfer functions to C_l.

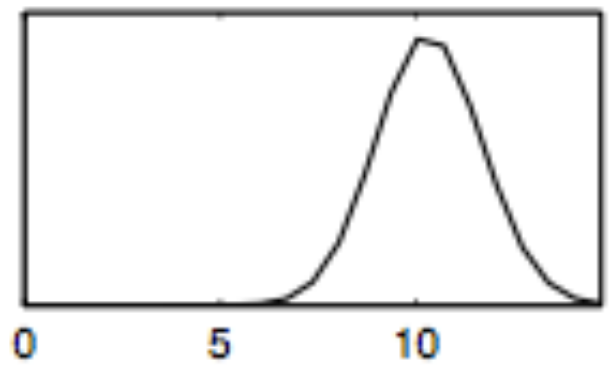
The damping scale

- The damping of the magnetic field is determined by the Alfven velocity and the Silk damping scale:

Subramanian & Barrow 1998
Jedamzik, Katalinic & Olinto 1998
Kahniashvili & Ratra 2006

$$k_D = (2.9 \times 10^4)^{\frac{1}{n_B+5}} \left(\frac{B_\lambda}{\mathrm{nG}}\right)^{\frac{-2}{n_B+5}} \left(\frac{2\pi}{\lambda|_{\mathrm{Mpc}}}\right)^{\frac{n_B+3}{n_B+5}} h^{\frac{1}{n_B+5}}$$

- This allows us to fix the damping scale by other cosmological and magnetic parameters and vary only B_λ and n_B



k_D [Mpc^{-1}]

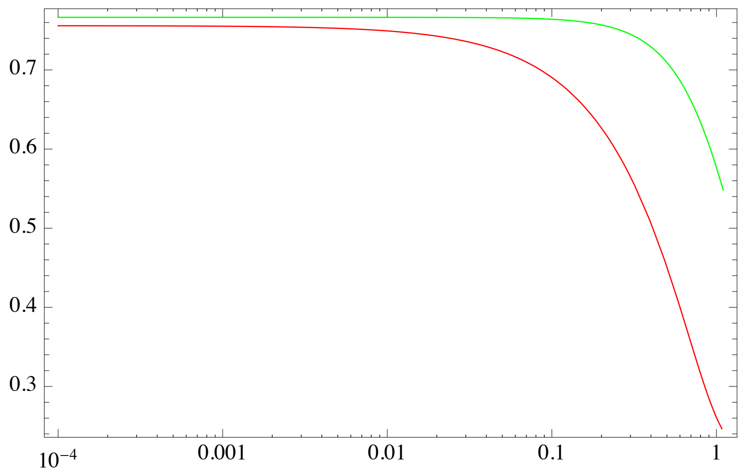
Paoletti & FF, 2012

Posterior probability for k_D [Mpc^{-1}] obtained from a Monte Carlo Markov Chain on the six cosmological parameters, B_λ and n_B plus nuisance foreground parameters. This posterior probability is marginalized over the other parameters varied in the Monte Carlo.

- Wavelengths relevant for CMB anisotropies are almost not affected by the quantitative details with which magnetic fields are damped by viscosity.

$|\rho_B(k)|^2$

$n_B = 2$



k/k_D

Damping modeled as a sharp cut-off

Damping modeled as a Gaussian smoothing

- Analytical integration feasible for generic spectral index only by a damping with a sharp cut-off.
- The methodology of computing the correlators analitically is therefore quite powerful for the investigation of probes for PMF on scales much smaller than those relevant for CMB.

Predictions of inhomogeneous PMFs on CMB anisotropies

- Vector Contribution

Subramanian & Barrow 1998
Seshadri & Subramanian 2001
Mack, Kahniashvili & Kosowsky 2002
Lewis 2004
Yamazaki et al. 2006
Paoletti, FF & Paci 2008
....

- Tensor Contribution

Durrer, Ferreira & Kahniashvili 2000
Mack, Kahniashvili & Kosowsky 2002
Caprini, Durrer & Kahniashvili 2004
Lewis 2004
Paoletti, FF & Paci 2008
......

- Scalar contribution

Adams et al., 1997
Koh & Lee 2002
Giovannini 2004, 2005, 2006 (2)
Kahniashvili & Ratra 2006
Yamazaki et al. 2006
Giovannini 2007
FF, Paci & Paoletti 2008
Paoletti, FF & Paci 2008
Giovannini & Kunze 2008 (6)
Giovannini 2009 (2)
......

Predictions

- Although several pioneering works were analytical, our approach is numerical.
- Several public codes which compute the evolution of linear fluctuations in the Einstein-Boltzmann system and compute the spectrum of CMB anisotropies and matter:

http://camb.info/ **Code for Anisotropies in the Microwave Background** Lewis & Challinor, 2000

http://www.thphys.uni-heidelberg.de/~robbers/cmbeasy/ Doran, Robbers, Mueller 2003

http://class-code.net/ the Cosmic Linear Anisotropy Solving System Lesgourgues 2011

The progenitors of the CAMB and CMBEASY, COSMICS (Ma & Bertschinger 1995) and CMBFAST (Seljak & Zaldarriaga) are no longer supported.

- To predict the effects of PMF on cosmological perturbations and CMB anisotropies:

Modifications of the Einstein-Boltzmann system of equations to include PMF contribution

Inclusion of power spectra for the EMT components of PMF

Computation of initial conditions for cosmological perturbations deep in the radiation era

- The results are from a modified version of the CAMB code created and mantained by Daniela Paoletti.

Scalar fluctuations

- Synchronous gauge (or longitudinal gauge, quite common choices):

$$ds^2 = a^2(\tau)\left[-d\tau^2 + (\delta_{ij} + h_{ij})\,dx^i dx^j\right] \qquad h_{0\mu} = 0$$
$$= a^2(\tau)\left[-d\tau^2(1+2\psi) + (1-2\phi)\delta_{ij} dx^i dx^j\right]$$

- By the decomposition of symmetric tensor, the two scalar potentials are h (the trace) and η (being 6η the potential for the traceless part):

$$k^2\eta - \frac{1}{2}\mathcal{H}\dot{h} = 4\pi G a^2(\Sigma_n\,\rho_n\delta_n + \rho_B)\,,$$
$$k^2\dot{\eta} = 4\pi G a^2 \Sigma_n(\rho_n + P_n)\theta_n\,,$$
$$\ddot{h} + 2\mathcal{H}\dot{h} - 2k^2\eta = -8\pi G a^2(\Sigma_n\, c^2_{s\,n}\rho_n\delta_n + \frac{\delta\rho_B}{3})\,,$$
$$\ddot{h} + 6\ddot{\eta} + 2\mathcal{H}(\dot{h} + 6\dot{\eta}) - 2k^2\eta = -24\pi G a^2[\Sigma_n(\rho_n + P_n)\sigma_n + \sigma_B]$$

n = 4 components (baryons, CDM, radiation, neutrinos)

- Scalar part of the Lorentz force: $\nabla^2 L^{(S)} \equiv \nabla_i L^i$

$$\nabla^2 L^{(S)} = \frac{1}{4\pi}\left[(\nabla_i B_j(\mathbf{x}))\nabla_j B_i(\mathbf{x}) - \frac{1}{2}\nabla^2(B_i(\mathbf{x})B^i(\mathbf{x}))\right]$$

- From the conservation of the PMF EMT: $\sigma_B = \dfrac{\rho_B}{3} + L_B$

Baryons velocity

- Lorentz force acting on baryons velocity scalar potential:

$$\dot{\theta}_b = -\mathcal{H}\theta_b + k^2 c_{sb}^2 \delta_b - k^2 \frac{L^{(S)}}{\rho_b}$$

- Eq. for photons velocity during tight coupling prior to recombination:

$$\begin{aligned}\dot{\theta}_\gamma &= k^2\Big(\frac{\delta_\gamma}{4} - \sigma_\gamma\Big) + a n_e \sigma_T (\theta_b - \theta_\gamma) \\ &= -R^{-1}\Big(\dot{\theta}_b + \mathcal{H}\theta_b - c_s^2 k^2 \delta_b + k^2 \frac{L}{\rho_b}\Big) + k^2\Big(\frac{\delta_\gamma}{4} - \sigma_\gamma\Big)\end{aligned}$$

Initial conditions: scalar perturbations

$$
\begin{aligned}
h(k,\tau) &= -\frac{3}{4}\Omega_B\omega\tau + \frac{9}{32}\Omega_B\omega^2\tau^2 \\
\eta(k,\tau) &= \frac{1}{8}\Omega_B\omega\tau - \frac{3\Omega_B\omega^2\tau^2}{64} + \frac{(-165L_B - 55\Omega_B + 28R_\nu\Omega_B)}{168(15+4R_\nu)}k^2\tau^2 \\
\delta_\gamma(k,\tau) &= -\Omega_B + \frac{\Omega_B\omega\tau}{2} - \frac{3\Omega_B\omega^2\tau^2}{16} - \frac{(3L_B + \Omega_B - R_\nu\Omega_B)}{6(-1+R_\nu)}k^2\tau^2 \\
\delta_\nu(k,\tau) &= -\Omega_B + \frac{\Omega_B\omega\tau}{2} - \frac{3\Omega_B\omega^2\tau^2}{16} - \frac{(3L_B + \Omega_B - R_\nu\Omega_B)}{6R_\nu}k^2\tau^2 \\
\delta_b(k,\tau) &= -3\frac{\Omega_B}{4} + \frac{3\Omega_B\omega\tau}{8} + \frac{1}{8}k^2\tau^2\Omega_B - \frac{9}{64}\Omega_B\omega^2\tau^2\omega^2 - \frac{3L_Bk^2\tau^2}{8(-1+R_\nu)} \\
\delta_c(k,\tau) &= -\frac{3\Omega_B}{4} + \frac{3\Omega_B\omega\tau}{8} - \frac{9}{64}\Omega_B\omega^2\tau^2 \\
\theta_\gamma(k,\tau) &= \frac{3L_Bk^2\tau}{4(-1+R_\nu)} - \frac{\Omega_B}{4}k^2\tau + \\
&\quad k^2\tau^2\Big(-\frac{9L_B(-1+R_c)\omega}{16(-1+R_\nu)^2} + \frac{(-4+R_\nu+3R_c)\omega\Omega_B}{16(-1+R_\nu)}\Big) \\
\theta_\nu(k,\tau) &= \frac{3L_Bk^2\tau}{4R_\nu} - \frac{k^2(-1+R_\nu)\Omega_B\tau}{4R_\nu} + \frac{1}{16}k^2\tau^2\omega\Omega_B \\
\theta_b(k,\tau) &= \frac{3L_Bk^2\tau}{4(-1+R_\nu)} - \frac{1}{4}\Omega_Bk^2\tau + \\
&\quad k^2\tau^2\Big(-\frac{9L_B(-1+R_c)\omega}{16(-1+R_\nu)^2} + \frac{(-4+R_\nu+3R_c)\omega\Omega]}{16(-1+R_\nu)} \\
\theta_c(k,\tau) &= 0 \\
\sigma_\nu(k,\tau) &= -\frac{3L_B+\Omega_B}{4R_\nu} + \frac{\Omega_Bk^2(55-28R_\nu)\tau^2}{56R_\nu(15+4R_\nu)} + \frac{165L_Bk^2\tau^2}{56R_\nu(15+4R_\nu)} \\
F_3(k,\tau) &= -\frac{3k\tau(3L_B+\Omega_B)}{14R_\nu} + \frac{165L_B + 55\Omega_B - 28R_\nu\Omega_B}{7(430R_\nu + 112R_\nu^2)}
\end{aligned}
$$

Terms proportional to PMF energy and Lorentz density is the regular inhomogeneous solution

a. with $\frac{\delta_i}{1+w_i} = \frac{\delta_j}{1+w_j}$

b. exhibits compensation $\delta_\gamma + \delta_\nu + \Omega_B \simeq 0$
(the same for pressures, velocities)

c. $\zeta \propto \Omega_B\,, L_B \times \mathcal{O}(k^2\tau^2)$, i.e. this inhomogeneous mode does not carry curvature on large scales and is regular at early times

d. subsequently in the matter era this inhomogeneous mode source curvature perturbation on large scales.

e. neutrino hierarchy needs to be closed to higher order wtr to the adiabatic homogeneous solution

f. other initial conditions for the inhomogeneous solution can be chosen (isocurvature modes can also be studied in presence of PMFs, Giovannini & Kunze 2008)

g. agrees with Shaw and Lewis 2009

Initial conditions for the radiation era

with matter corrections

FF, Paci, Paoletti 2008
Paoletti, FF, Paci 2009
Paoletti, FF 2010

$$\Omega_B \equiv \frac{\rho_B}{\rho_\gamma + \rho_\nu} \qquad L_B \equiv \frac{L}{\rho_\gamma + \rho_\nu}$$

$$R_\nu = \frac{\rho_\nu}{\rho_\nu + \rho_\gamma} \qquad \omega = \frac{\Omega_m H_0}{\sqrt{\Omega_\nu + \Omega_\gamma}} \qquad R_c = \frac{\Omega_c}{\Omega_b + \Omega_c}$$

Vector fluctuations

- In absence of inhomogeneous sources or free streaming particles, vector fluctuations are usually neglected since corresponds to decaying modes. In presence of PMF vector modes play a relevant role.

$$\dot{h}_i^V + 2\mathcal{H}h_i^V = -16\pi G a^2 \left[\Pi_{\nu\, i}^{(V)} + \Pi_{\gamma\, i}^{(V)} + \Pi_{B\, i}^{(V)}\right]/k$$

- As for the scalar sector the anisotropic stress in related to the Lorentz force.

$$L_i^{(V)} = k\Pi_i^{(V)}$$

- The vector part of the velocity field of baryons is driven by the (vector part of the) Lorentz force

$$\dot{v}_{b\, i} + \mathcal{H}v_{b\, i} = -\frac{\rho_\gamma}{\rho_b}\left(\frac{4}{3}n_e a\sigma_T(v_{b\, i} - v_{\gamma\, i}) - \frac{L_i^V}{\rho_\gamma}\right)$$

Tensor fluctuations

- Tensor perturbations:

$$\ddot{h}_{ij} + 2\mathcal{H}\dot{h}_{ij} + k^2 h_{ij} = 16\pi G a^2(\rho_\nu \pi^\nu_{ij} + \Pi^{(B,T)}_{ij})$$

- The neutrino tensor anisotropic stress satisfy a differential equation of the Boltzmann hierarchy. Deep in the radiation era:

$$\ddot{h}^T_k + \frac{2}{\tau}\dot{h}^T_k + k^2 h^T_k = \frac{6}{\tau^2}[R_\nu \sigma^{(T)}_\nu + (1-R_\nu)\tilde{\Pi}^{(T)}_B]$$

$$\dot{\sigma}^{(T)}_\nu = -\frac{4}{15}\dot{h}_k - \frac{k}{3}J_3$$

$$\dot{J}_3 = \frac{3}{7}k\sigma^{(T)}_\nu$$

$$J_4 = 0$$

- Initial conditions:

$$h_k = \frac{15(1-R_\nu)\tilde{\Pi}^{(T)}_B (k\tau)^2}{56(15+4R_\nu)}$$

$$\sigma^{(T)}_\nu = -\frac{(1-R_\nu)}{R_\nu}\tilde{\Pi}^{(T)}_B\left[1 - \frac{15(k\tau)^2}{14(15+4R_\nu)}\right]$$

- The absence of a constant mode on large scales is the consequence of compensation - as for the scalar mode - for the regular tensor inhomogeneous solution in presence of PMF and free streaming neutrinos.

CMB anisotropies

- Photons travelling from the surface of last scattering surface when matter and radiation decoupled.
- Anisotropies of the cosmic black-body radiation with T=2.7255 K

$$\frac{\delta T}{T}(\theta,\phi) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\theta,\phi)$$

$$C_\ell = \sum_\ell \frac{|a_{\ell m}|^2}{2\ell+1} \qquad \langle a_{\ell' m'} a_{\ell m} \rangle = C_\ell \delta_{\ell\ell'} \delta_{mm'}$$

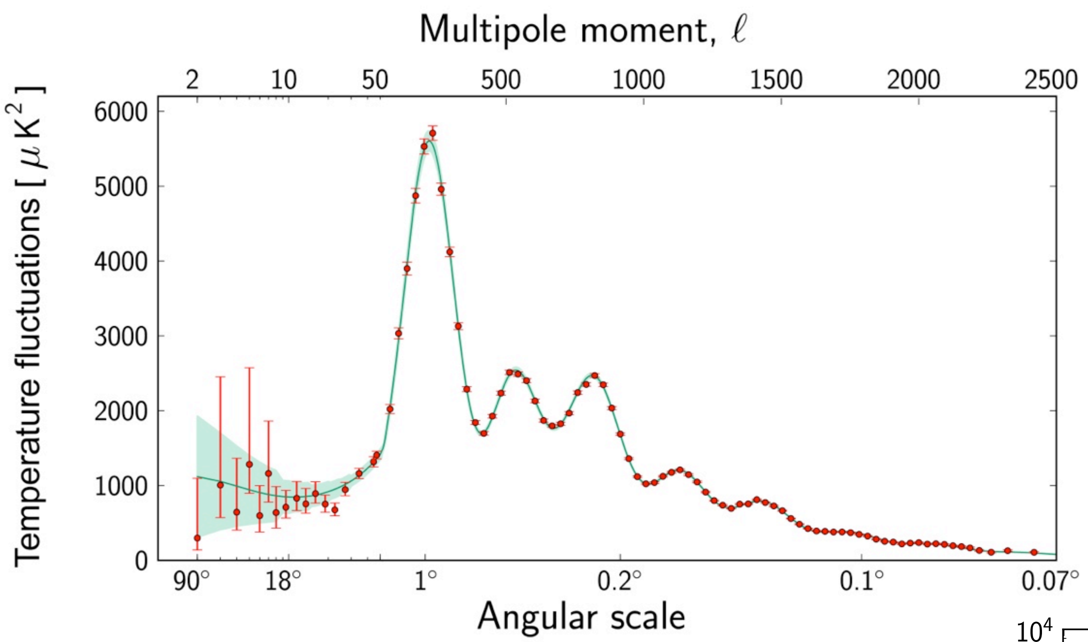
- The anisotropy spectrum is a convolution of the 3-D Fourier spectrum

$$C_\ell = 4\pi \int d\ln k \left(k^3 \frac{|\mathcal{R}(k)|^2}{2\pi^2} \right) \left[\left(\frac{\delta_\gamma}{4} + \psi \right)(\tau_e, k) j_\ell(k\Delta\tau) - v_b(\tau_e, k) j'_\ell(k\Delta\tau) + \int_{\tau_e}^{\tau_o} (\psi' + \phi') d\bar{\tau} \right]^2$$

Sachs-Wolfe term — Doppler term — Integrated Sachs-Wolfe term

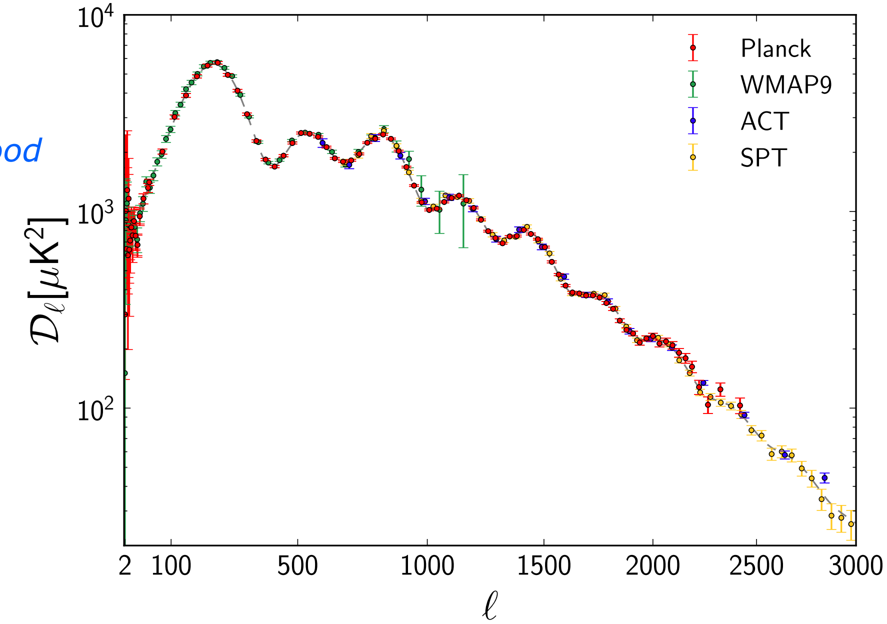
- Analogous formula for vector and tensor contribution to CMB anisotropies

Status of CMB temperature measurements



Planck 2013 results XXV: CMB power spectra and likelihood

Planck 2013 results XXVI: Cosmological parameters



CMB polarization

- Thomson scattering generates partial linear polarization from density perturbations close to the last scattering surface

- Two additional observables:

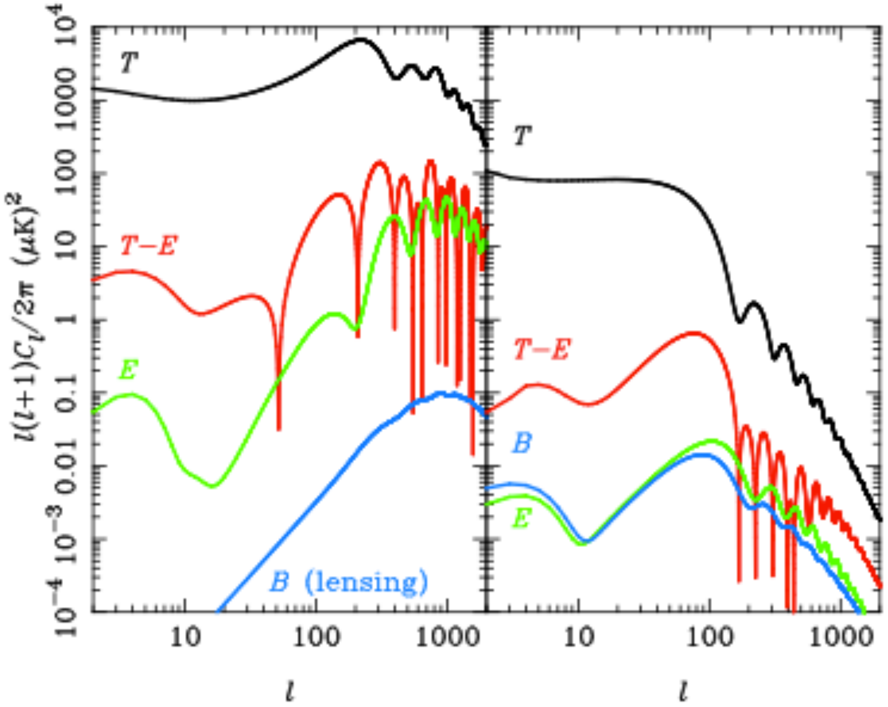
  Stokes parameters Q, U at the map level

  E,B decomposition at the power spectrum

  level

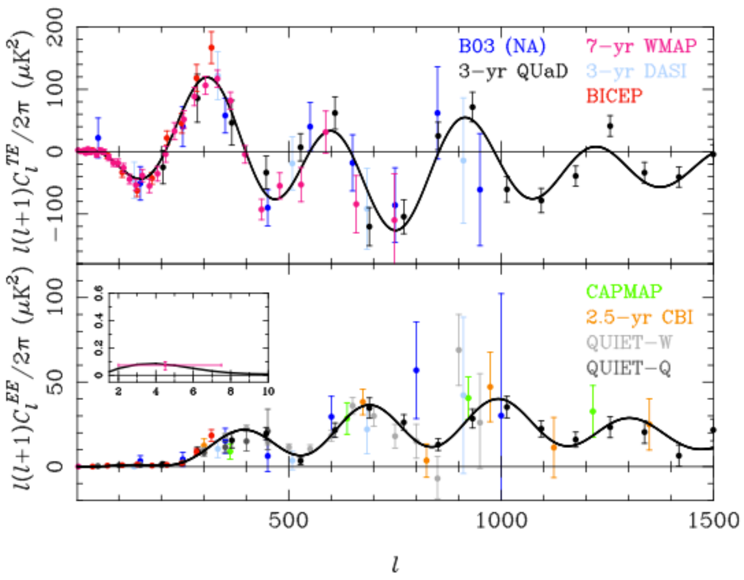
- B modes generated only by vector and tensor fluctuations as primary anisotropies or by lensing of scalar fluctuations as a secondary anisotropy

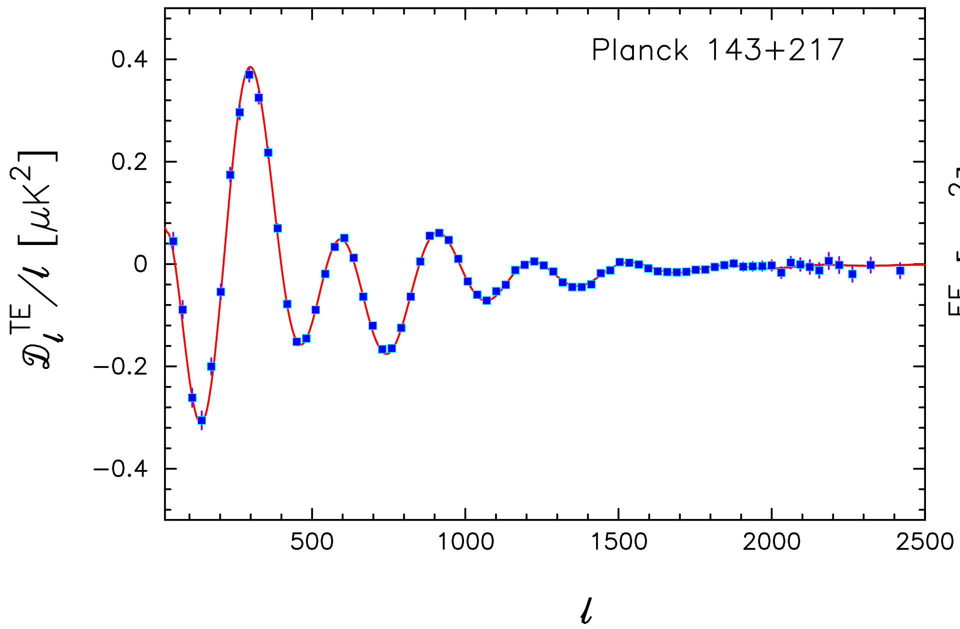
- Only cross-correlation between T and E for parity symmetry

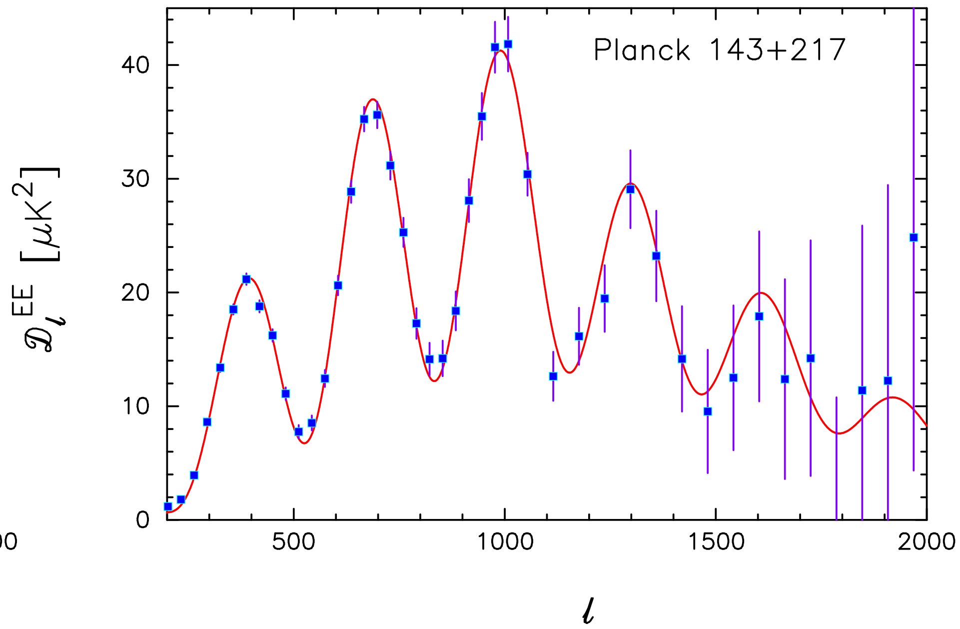


Courtesy from A. Challinor

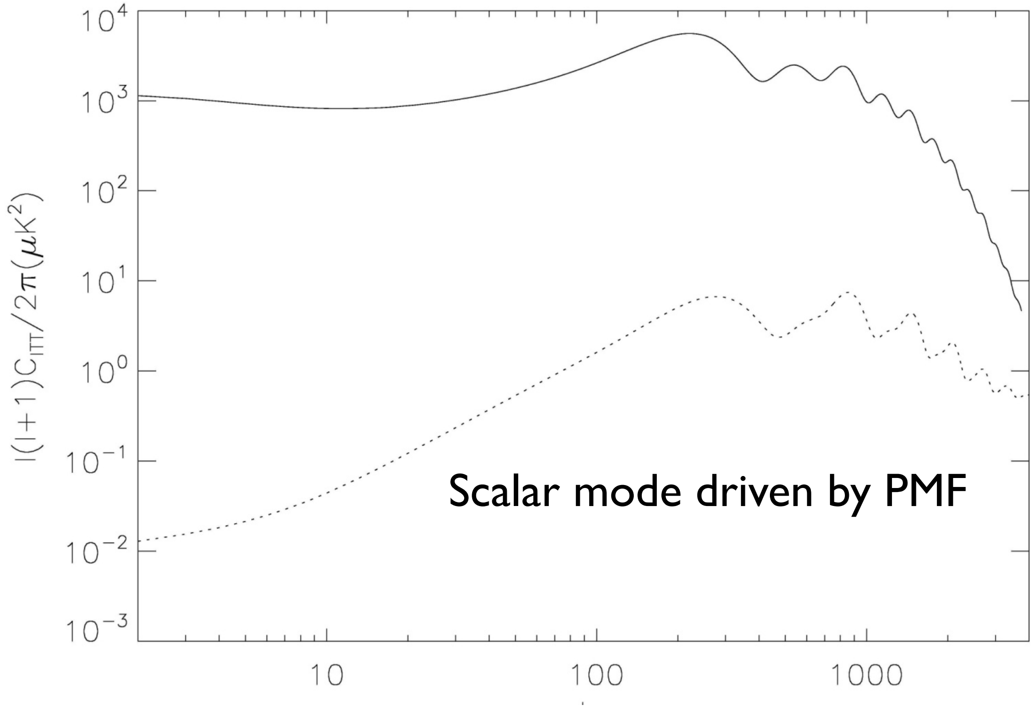
Status of CMB polarization measurements



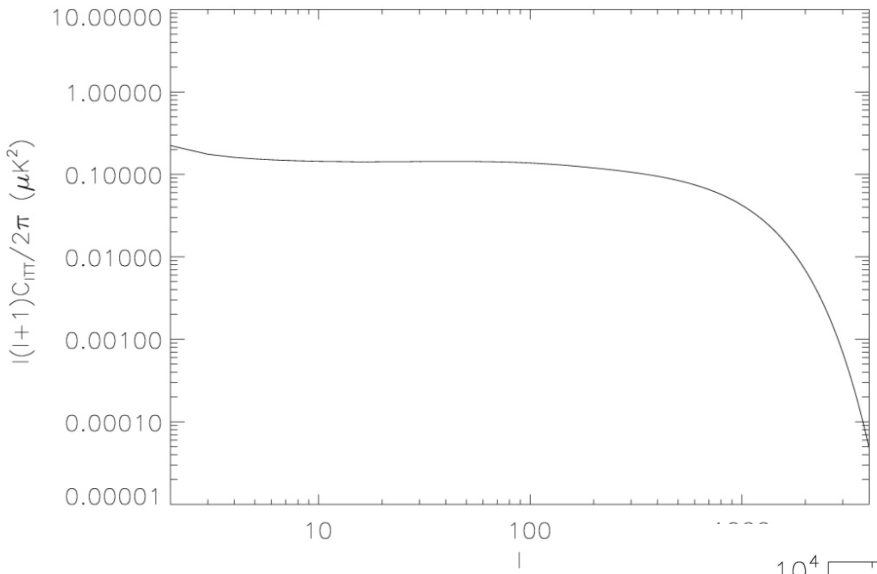




PMF scalar contribution

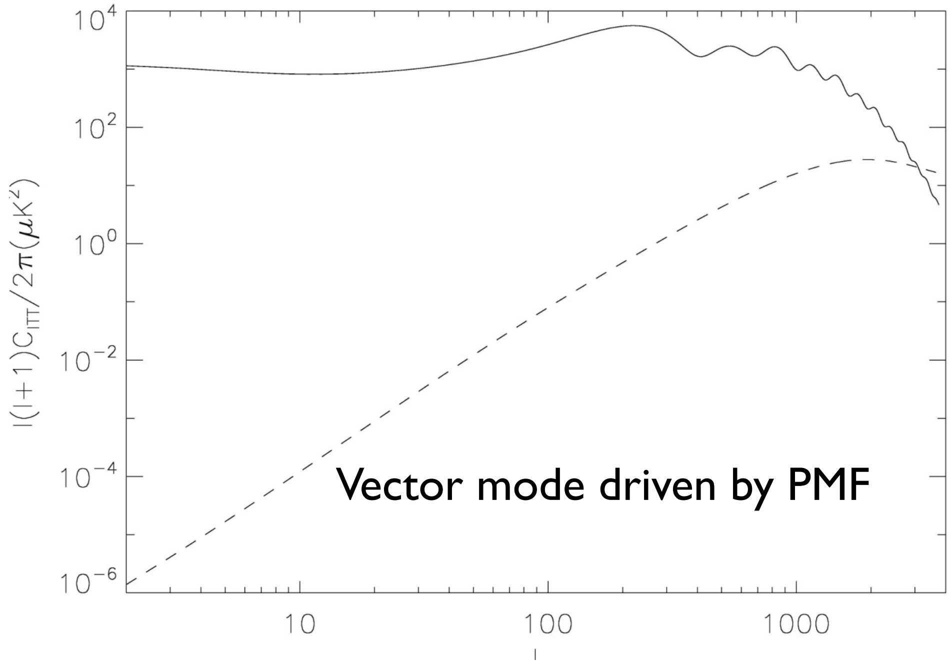


PMF vector contribution



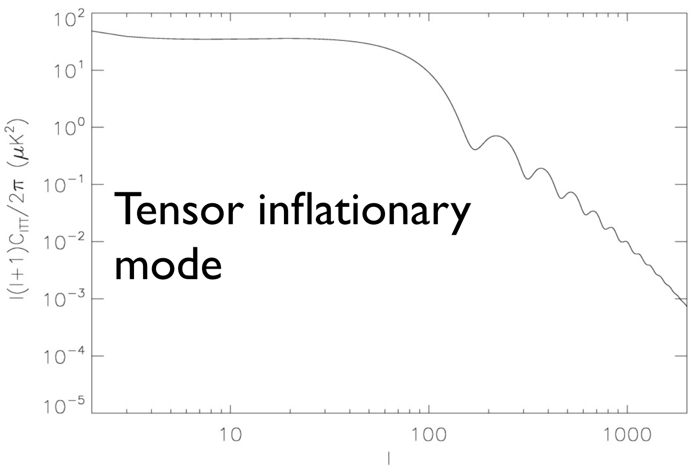
Lewis 2003

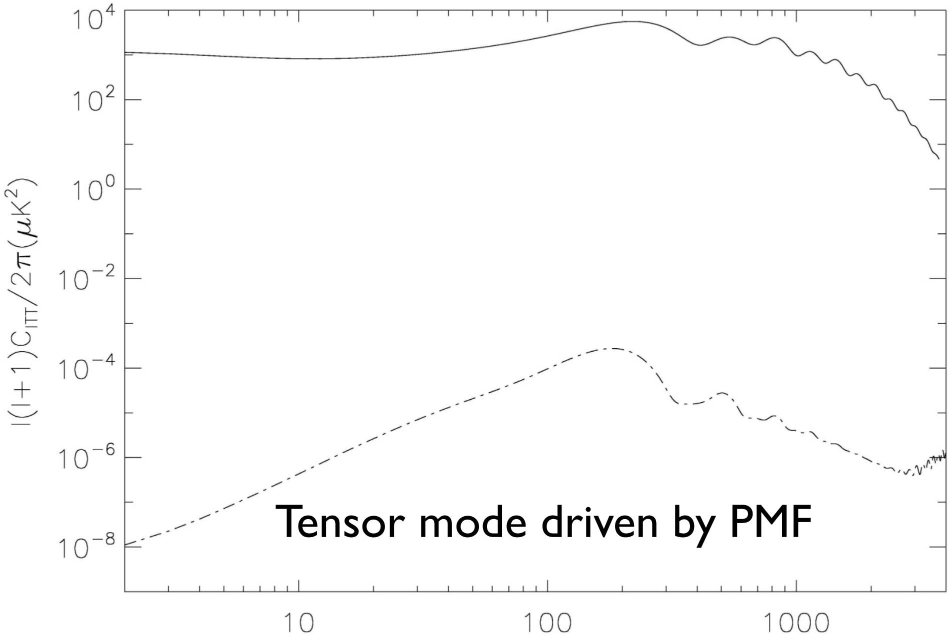
Vector mode driven by neutrino hierarchy



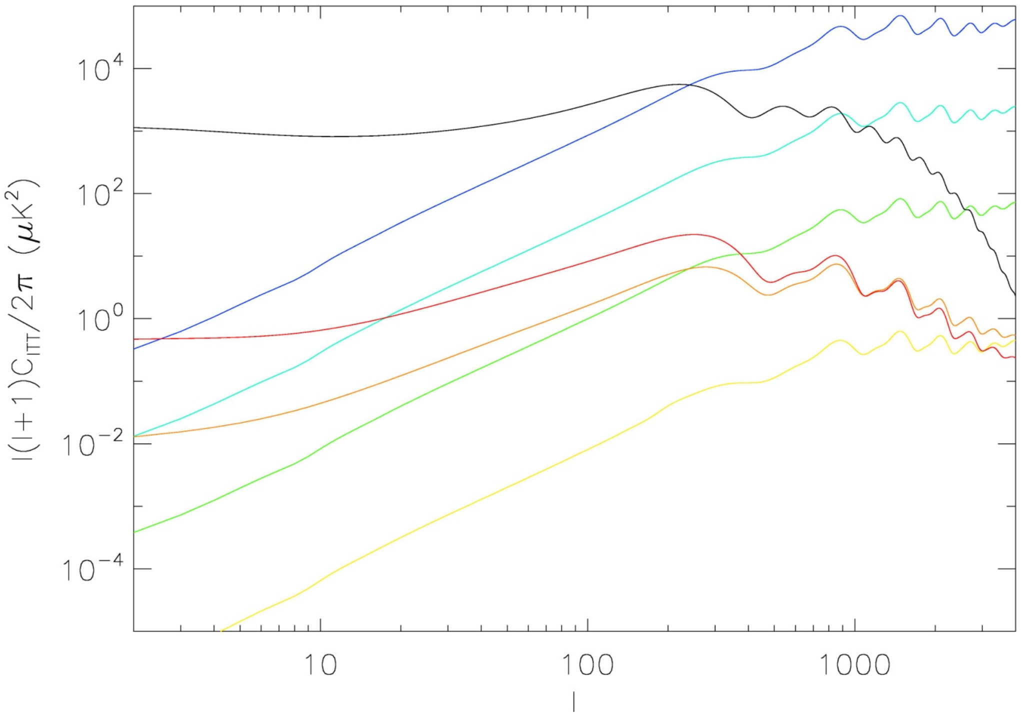
Vector mode driven by PMF

PMF tensor contribution





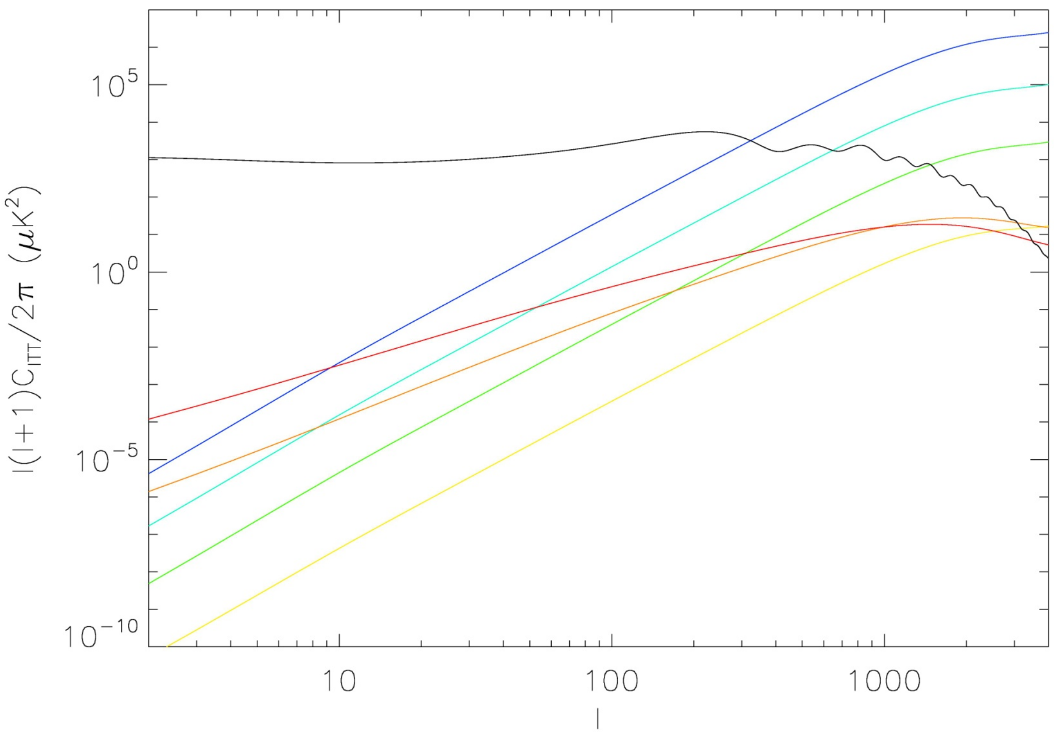
Dependence on the spectral index: scalar



Black Primary CMB
Blue n=2
Cyan n=1
Green n=0
Yellow n=-1.5
Orange n=-2.5
Red n=-2.9

- For k << k_D and n_B>-3/2 the Fourier spectra of the scalar part of the PMF EMT have a white noise spectrum and the same occurs for the scalar contribution to CMB anisotropies.

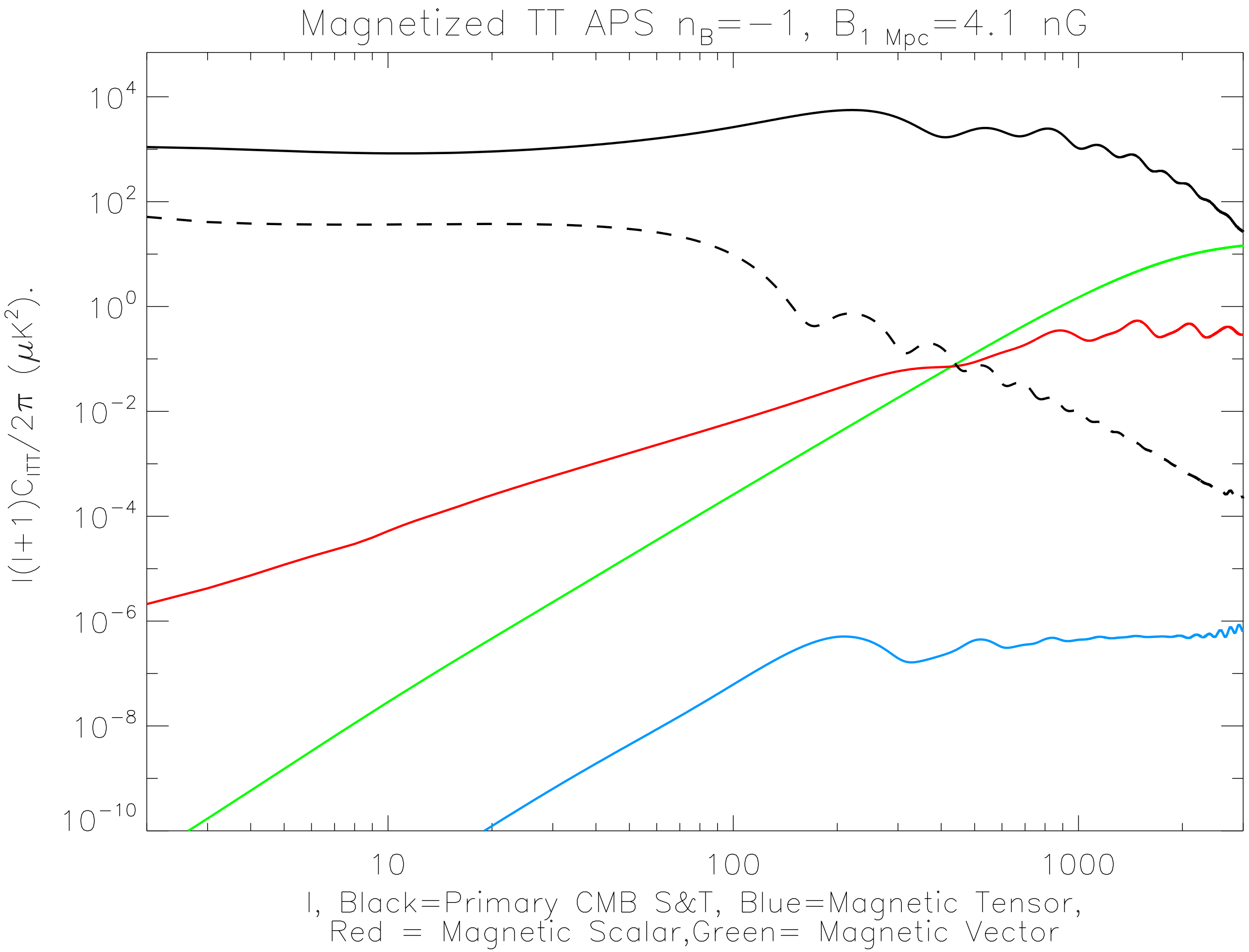
Dependence on the spectral index: vector



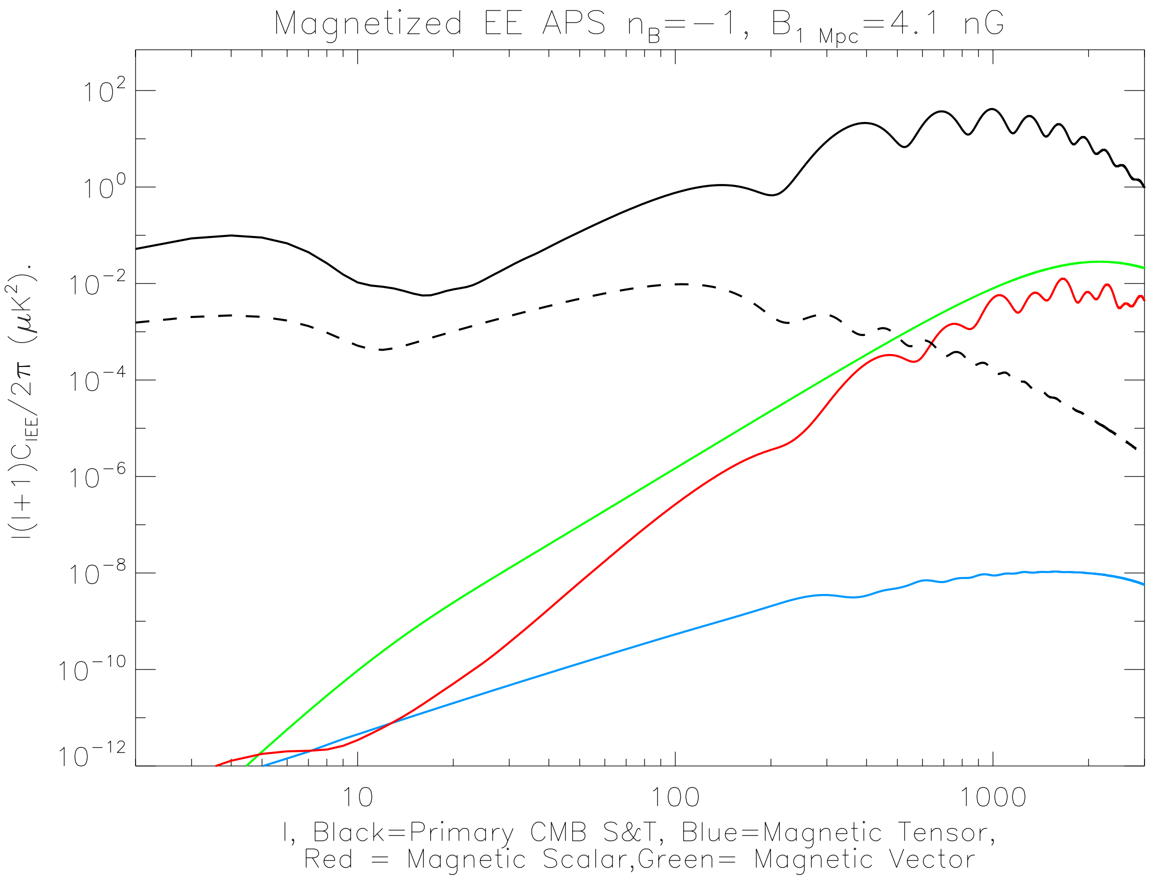
Black Primary CMB
Blue n=2
Cyan n=1
Green n=0
Yellow n=-1.5
Orange n=-2.5
Red n=-2.9

- For k << k_D and n_B>-3/2 the Fourier spectra of the vector part of the PMF EMT have a white noise spectrum and the same occurs for the vector contribution to CMB anisotropies.

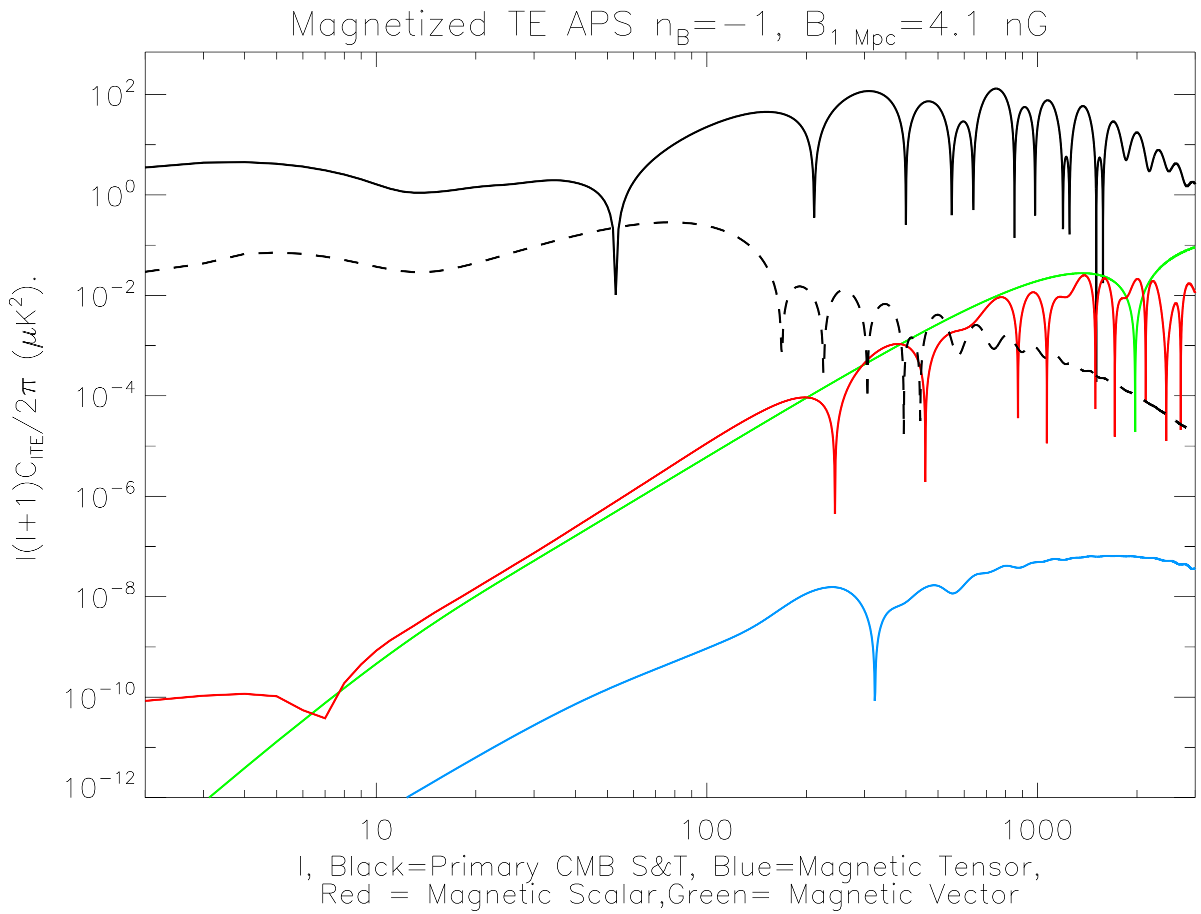
Tutti insieme: TT



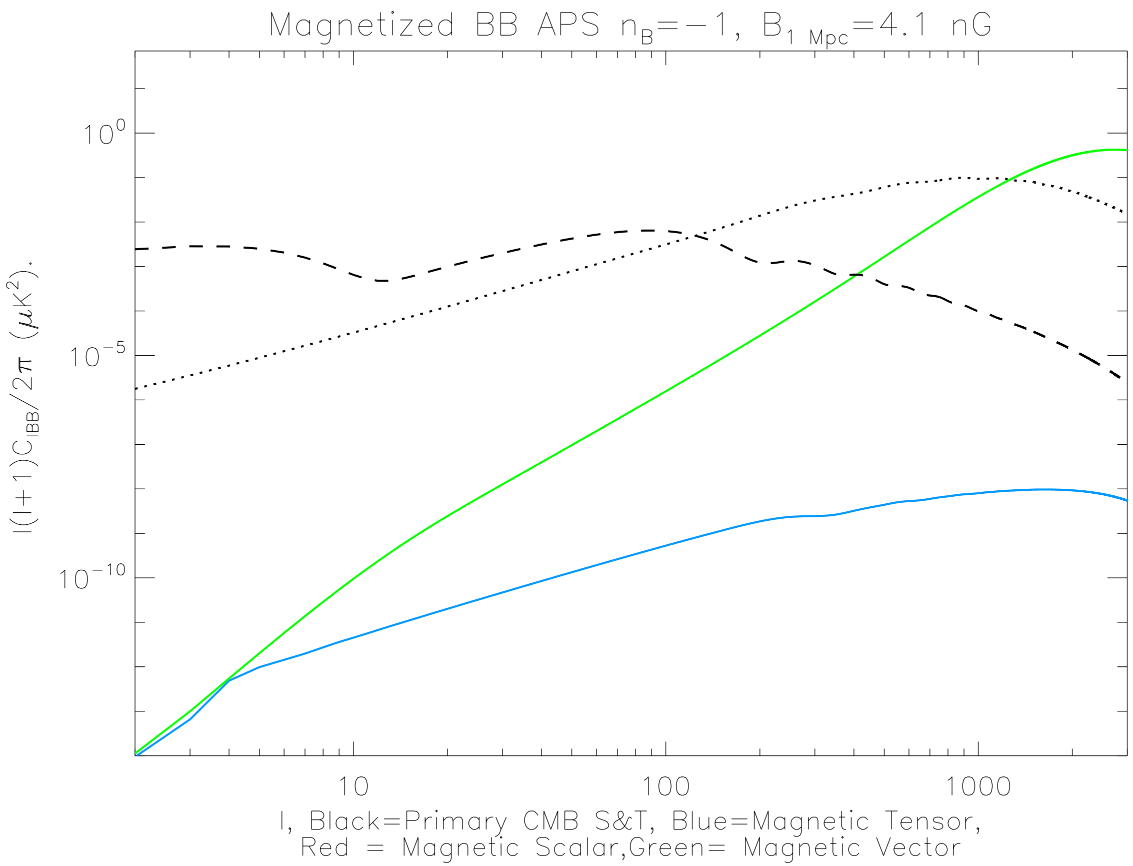
Tutti insieme: EE



Tutti insieme: TE



Tutti insieme: BB



Comparison of CMB generated by compensated and passive modes

- So far we have shown results based on initial conditions which satisfy the Einstein-Boltzmann system after neutrino decoupling and correspond to cosmological perturbations which are regular (these perturbation carry an infinitesimal curvature on long wavelength as for the so-called isocurvature modes)

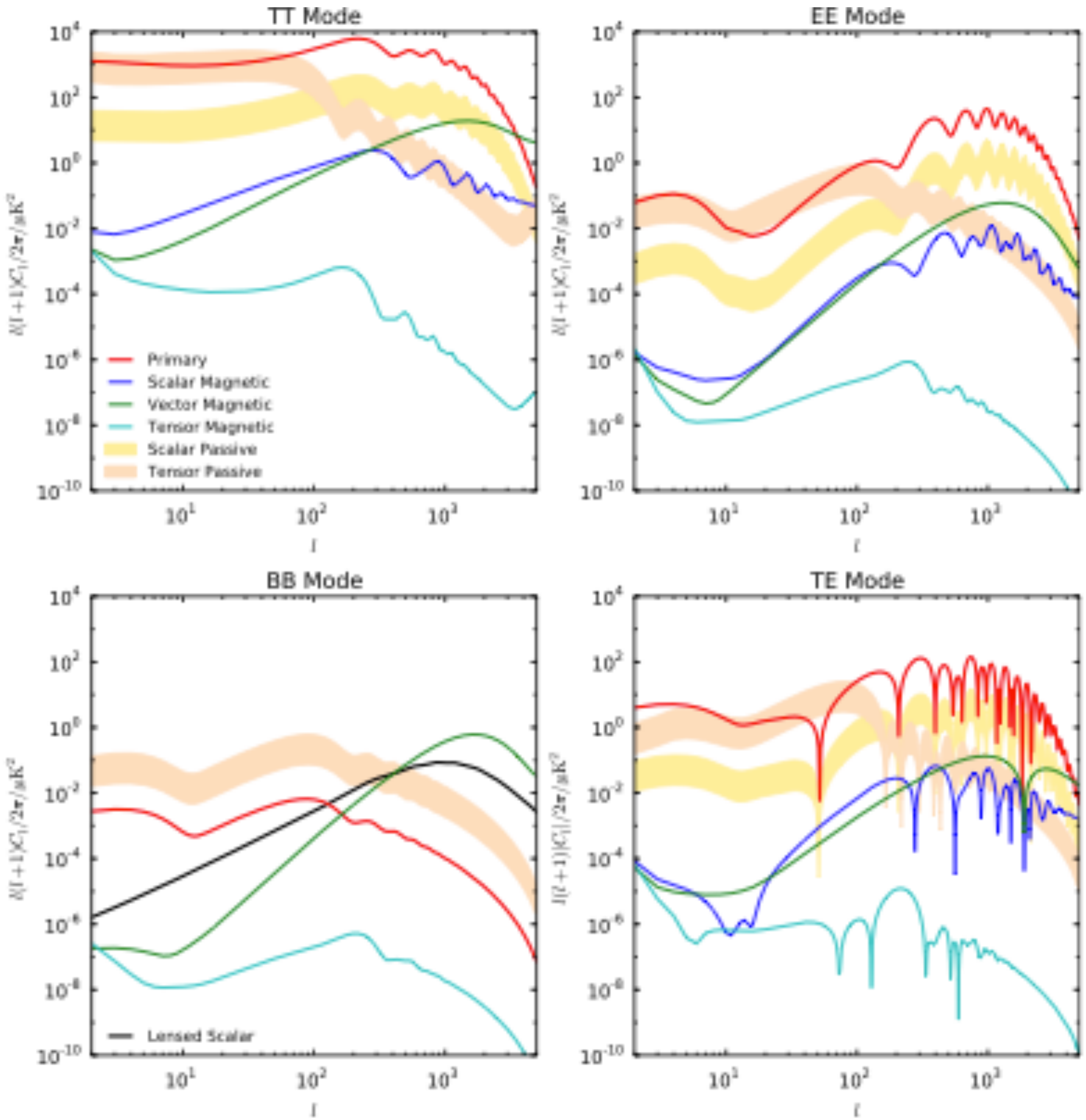
- It has been argued that **passive** scalar and tensor fluctuations are generated at neutrino decoupling by the matching of pre-existing scalar and tensor fluctuations (which are characterized by modes which are singular and not regular before neutrino decoupling) which were not compensated by the free streaming neutrinos.

- These passive modes correspond to adiabatic scalar and tensor homogeneous solution of the Einstein-Boltzmann system with an amplitude fixed by the spectrum of the PMF

$$\mathcal{R} \simeq \mathcal{R}(\tau_B) - \frac{3}{2}\sigma_B \left[\ln\left(\frac{\tau_\nu}{\tau_B}\right) + \left(\frac{5}{8R_\nu} - 1\right) \right]$$

$$h \simeq \Pi_B^{(T)} \left[\ln\left(\frac{\tau_\nu}{\tau_B}\right) + \left(\frac{5}{8R_\nu} - 1\right) \right]$$

Shaw and Lewis 2010



Shaw and Lewis 2010

- Interesting results which deserve further attention (including a careful matching through the neutrino decoupling process)

The impact of helicity

Ballardini, FF, Paoletti 2013

- An helical component of a SB of PMF would open a new window on the physics of early Universe or in our understanding of turbulence.

$$\langle B_i(\mathbf{k}) B_j^*(\mathbf{k}')\rangle = \frac{(2\pi)^3}{2}\delta(\mathbf{k}-\mathbf{k}')\Big[(\delta_{ij}-\hat{k}_i\hat{k}_j)P_S(k) + i\epsilon_{ijl}\hat{k}_l P_A(k)\Big]$$

- P_A contributes to all the PMF EMT Fourier correlations shown so far (parity even correlations). As an example consider the tensor anisotropic stress

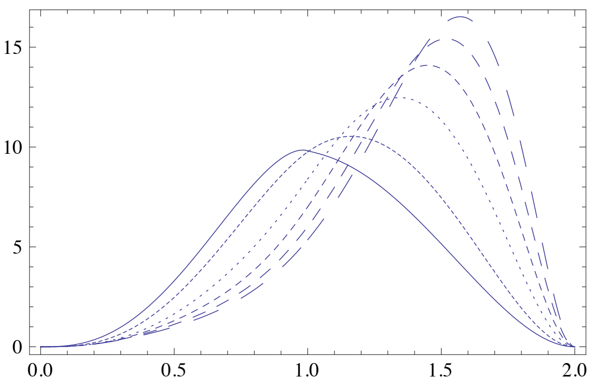
$$|\Pi^{(T)}(k)|^2 = \frac{2}{(4\pi)^5}\int d^3p\Big[P_S(p)P_S(|\mathbf{k}-\mathbf{p}|)(1+\gamma^2)(1+\beta^2) + 4P_A(p)P_A(|\mathbf{k}-\mathbf{p}|)\gamma\beta\Big]$$

Caprini, Durrer, Kahni

$k^3|\Pi^{(T)}(k)|^2$

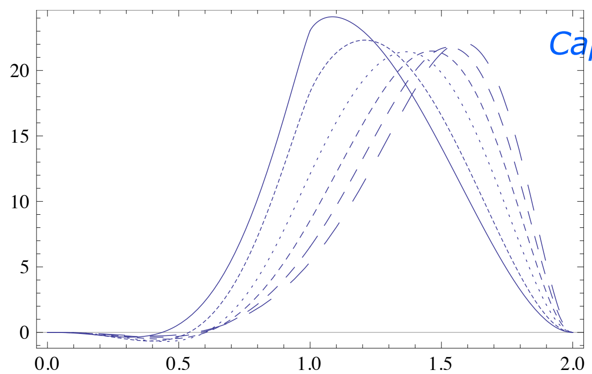
$n_S=n_A=-3/2, -1,0,1,2,3$

from the solid to large dashed



k/k_D

$$\propto \int d\mathbf{p}\, P_S(k)P_S(|\mathbf{k}-\mathbf{p}|)$$



k/k_D

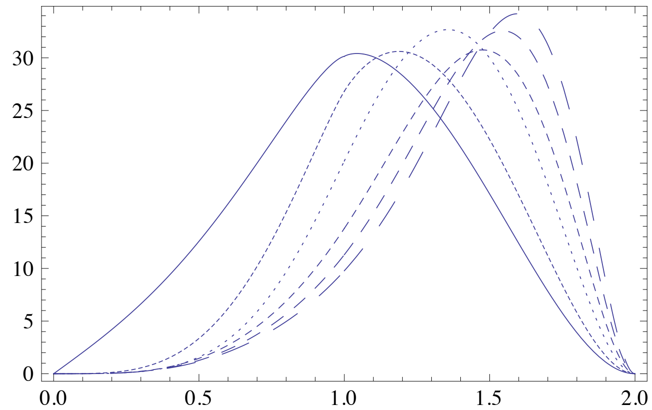
$$\propto \int d\mathbf{p}\, P_A(k)P_A(|\mathbf{k}-\mathbf{p}|)$$

Plot which shows the relative magnitude in the case of a maximally helical background: $A_S=A_A$, $n_S=n_A$

- A non-zero helical component generates also correlations which do not respect parity symmetry (parity odd). As an example consider the tensor part:

$$A^{(T)}(k) = \frac{4}{(4\pi)^5} \int d^3p \left[P_S(p) P_A(|\mathbf{k}-\mathbf{p}|)(1+\gamma^2)\beta + P_A(p) P_S(|\mathbf{k}-\mathbf{p}|)\gamma(1+\beta^2) \right]$$

$k^3 A^{(T)}(k)$



k/k_D

n_B=-3/2,-1,0,1,2,3

from the solid to large dashed

- The role of exact computation of the angular part is emphasized in comparison to previous works.

Caprini, Durrer, Kahniashvili 2004