Primordial magnetic fields in the Early Universe and anisotropies of the cosmic microwave background

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Outline: I part

- Brief introduction of cosmology
- Primordial magnetic fields in the early Universe
- A (non-exhaustive) review of mechanisms for the amplification of gauge fields during inflation

Outline: II part

- Cosmological perturbations in presence of a stochastic background (SB) of primordial magnetic fields (PMF)
- The energy-momentum tensor (EMT) for the SB of PMF
- The spectrum of CMB anisotropies in intensity and polarization generated by a SB of PMF
- Constraints on PMF from the CMB power spectrum
- PMF non-gaussianities in the CMB pattern
- PMF impact on matter power spectrum
- PMF impact on spectral distortions

References: reviews

D. Grasso and H.R. Rubinstein, Physics Reports 348 (2001) 161

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K. Subramanian, Astronomische Nachrichten 331 (2010) 110

A. Kandus, K. Kunze, C. Tsagas, Physics Reports 505 (2011) 1

D. G. Yamazaki et al., Physics Reports 517 (2012) 141

R. Durrer and A. Neronov, arXiv:1303.7121, to appear in Astronomy & Astrophysics Review (2013)

Cosmological Magnetic Fields

- The origin of the large scale magnetic fields observed in galaxies and clusters of galaxies is an open issue of great importance in modern astrophysics.
- These lectures are about cosmological primordial magnetic fields which are frozen in prior to cosmological recombination (i.e. z = 1100), however after nucleosynthesis.
- A primordial hypothesis for generating magnetic seeds (which can be amplified afterwards by adiabatic compression) can be connected to observations of strong magnetic fields in galaxies at high redshift,
- Recent observations by FERMI can be interpreted by large scale magnetic fields in voids with a lower limit on the amplitude around 10^{-15}-10^{-18} G.
- Nucleosynthesis constrains the magnitude of primordial magnetic field to less than 300 nG (3 muG would correspond to the same radiation energy density)
- Cosmological primordial magnetic fields leave imprints on a host of observables as the CMB pattern, LSS, etc ...

Bernet et al. 2008
Wolfe et al. 2008

Neronov & Vovk 2010
Tavecchio et al. 2010
Taylor, Vovk & Neronov 2011
Vovk et al. 2012
Neronov 2013

Grasso & Rubinstein1995

Standard Big Bang Cosmology

The cosmological principles states that the Universe on large scales is homogeneous and isotropic. In the context of General Relativity this implies that the expanding space-time is described by the Robertson-Walker (RW) metric (k being the curvature of the spatial sections)

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-k\,r^2} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right]$$

where the scale factor obeys to the Friedman RW

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{\rho}{3M_{\rm pl}^2} \qquad\qquad H \equiv \frac{\dot{a}}{a}$$

$$M_{\rm pl}^{-2} = 8\pi G$$

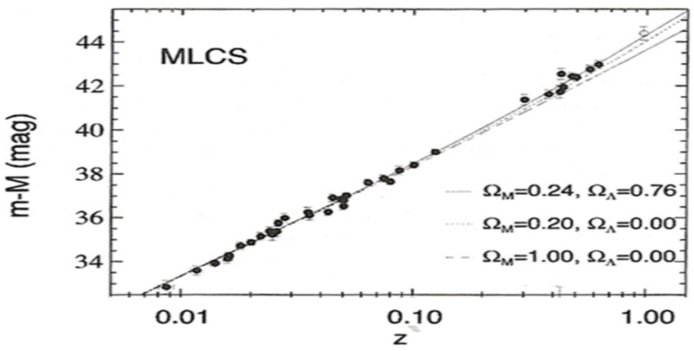
$$\frac{\ddot{a}}{a} = -\frac{\rho + 3p}{6M_{\rm pl}^2} \qquad\qquad \frac{1}{1+z} = \frac{a(t)}{a(t_0)}$$

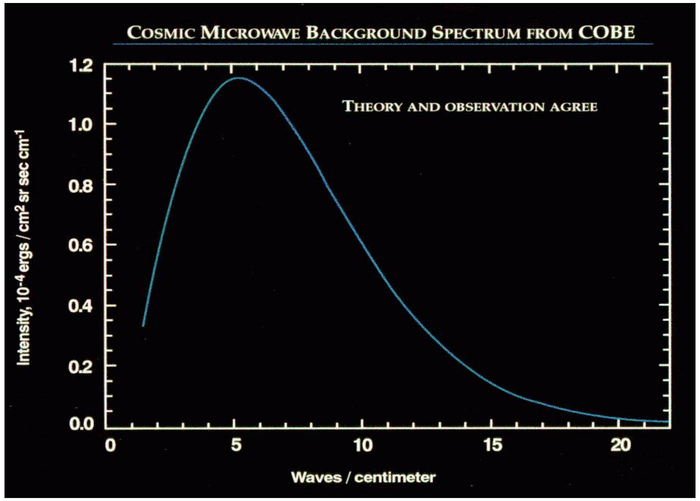
with ρ and p as a sum of N perfect fluids with barotropic equation of state ($p_i = w_i \rho_i$ with w_i as time-independent parameter). From the energy-momentum conservation of the fluids (redundant with the above second equation)

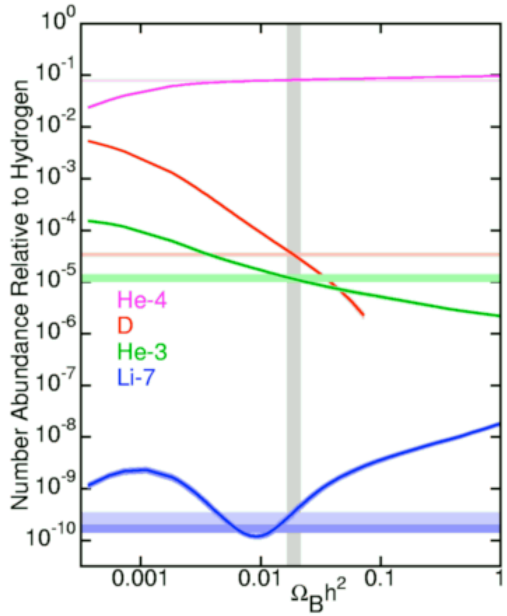
$$\dot{\rho}_i + 3H(\rho_i + p_i) = 0 \quad \rightarrow \quad \rho = \frac{\rho_{i\,0}}{a^{3(1+w_i)}}$$

For a flat universe (k=0), considering a single fluid: $a(t) = a_0 t^{\frac{2}{3(1+w)}}$

Standard Big Bang cosmology with an energy density dominated by radiation and matter has 3 important cosmological evidences: the Hubble law, the existence of the nearly isotropic and nearly black-body Cosmic Microwave Background (CMB) and the abundance of light elements.







Problems of the Standard Big Bang Cosmology

Standard Big Bang cosmology also faces problems which do not find a simple explanations within the above assumptions.

Horizon Problem

The size of the causally connected distance on the LSS of CMB $(z \simeq 1100)$

$$d_H(t_i, t_{\rm rec}) = a(t_{\rm rec}) \int_{t_i}^{t_{\rm rec}} \frac{dt'}{a(t')}$$

is seen now under an angular scale of a degree. If this was the maximum distance under which microphysics forces would have always operated, we do not have the explanation of why CMB is isotropic on scale much larger than a degree scale.

Flatness Problem

For a FRW universe with matter satisfying $\rho + 3p > 0,\ \ \Omega = 1$, with Ω being the critical density $\Omega = \frac{\rho}{3M_{\rm pl}^2 H^2} = 1 + \frac{k}{a^2 H^2}$ is an unstable fixed point for an expanding universe:

$$\frac{d(\Omega - 1)}{dt} = (\Omega - 1)\frac{\rho + 3p}{3HM_{\rm pl}^2}$$

Since $\Omega(t_0) \sim \mathcal{O}(1)$, $|\Omega(t) - 1|$ should be extremely fine tuned to zero in the past when $\rho + 3p > 0$

Monopole Problem

Monopoles and other stable relics are predicted in many particle physics models at high energies/temperatures. The energy density of these relics might be not negligible today, whereas these relics are not observed.

Structure Formation Problem

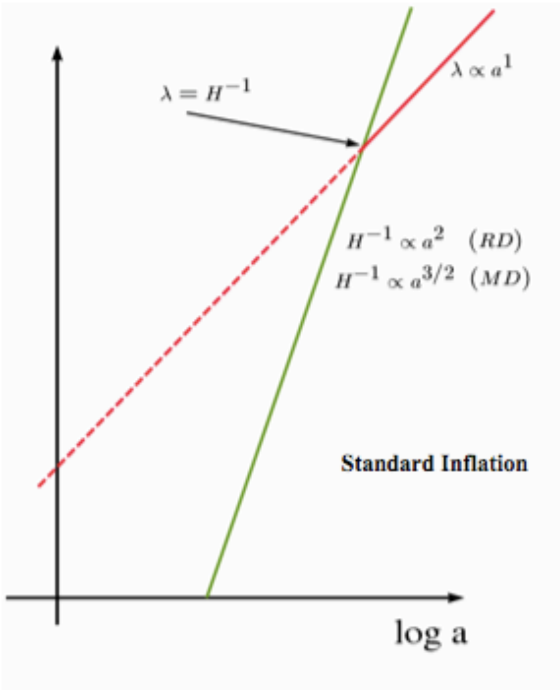
Since 70's Harrison and Zeldovich recognized that a scale invariant spectrum for gravitational perturbations was in agreement with observations.

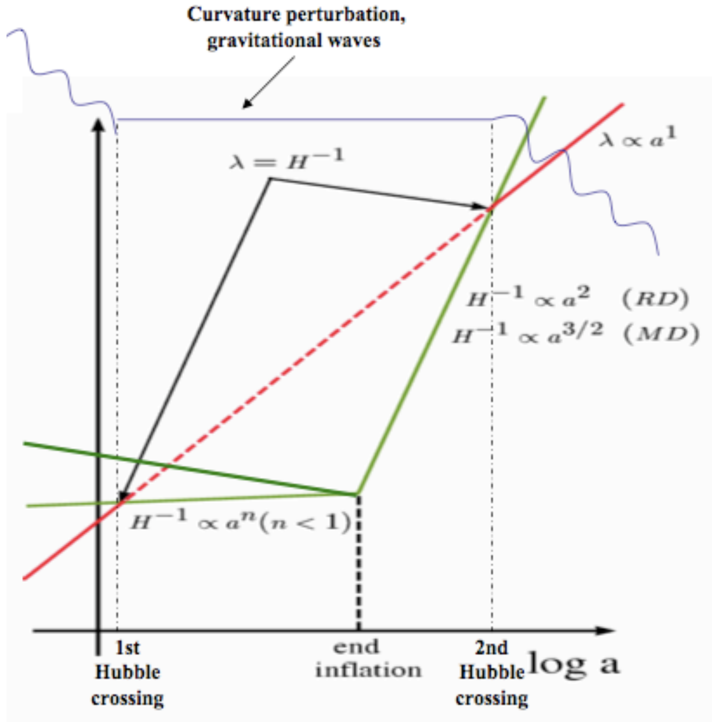
$$k^3 |\mathcal{R}(k)|^2 \propto k^{n_{\rm s}-1}$$

For microscopic causal mechanisms and gravity alone it is difficult to explain such a spectrum.

Introducing Inflation

All these issue are addressed within the inflationary scenario: an accelerated stage in the early universe is an elegant way to address directly the horizon, flatness and monopole problems. We will also see how it can solve also the structure formation problem in a sufficient general way.





As it was clear from the FRW equations, acceleration requires $\rho + 3p < 0$, and the simplest hypothesis of constant acceleration requires a cosmological constant or a matter component with $p \simeq -\rho$.

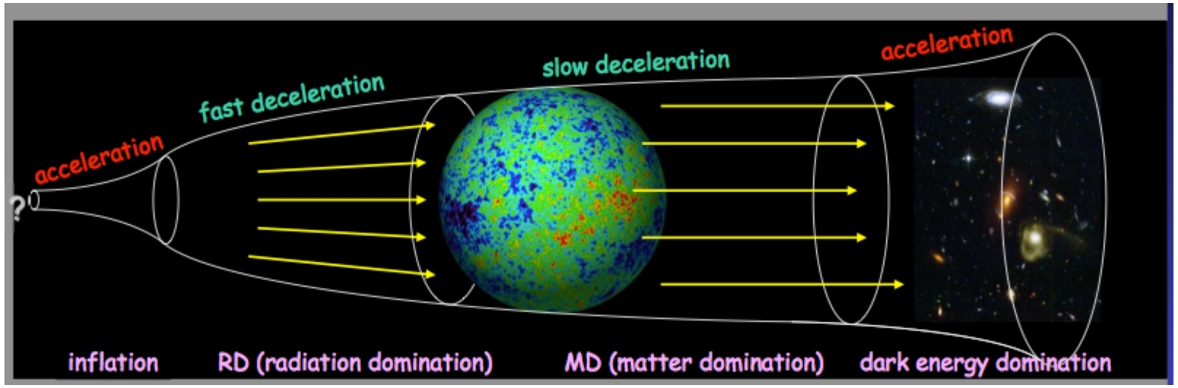
$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{\rho}{3\,M_{\rm pl}^2} + \frac{\Lambda}{3}$$

$$\frac{\ddot{a}}{a} = -\frac{(\rho + 3p)}{6M_{\rm pl}^2} - \frac{2}{3}\Lambda$$

A cosmological constant, which can be interpreted as an homogeneous fluid with constant energy density and $p_\Lambda = -\rho_\Lambda$, tends to dominate over ordinary matter and curvature. A Universe with exponential expansion (de Sitter solution, $a = a_0 e^{\sqrt{\frac{\Lambda}{3}}t}$) is not only a stable solution among homogeneous cosmologies, but is also a global attractor among inhomogeneous cosmologies.

An simple cosmological constant may therefore solve the problem of standard big bang cosmology, but it would prevent a simple connection to a radiation dominated universe.

A much smaller cosmological constant is the simplest solution for the present acceleration of the Universe.



Inflation as Scalar Fields Dynamics

A simple cosmological constant may therefore solve the problem of standard big bang cosmology, but it would prevent a simple connection to a radiation dominated universe.

Scalar fields are a matter source described by:

$$\mathcal{L} = -\frac{g_{\mu\nu}}{2}\partial_\mu\phi\partial_\nu\phi - V(\phi) \qquad T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\mathcal{L}$$

By writing an inhomogeneous scalar field in RW metric:

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi) + \frac{(\nabla\phi)^2}{2a^2}$$

$$p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi) - \frac{(\nabla\phi)^2}{6a^2}$$

a. if the kinetic term dominates the scalar contribution behaves as stiff matter with $p_\phi \simeq \rho_\phi \;\rightarrow\; w_\phi \simeq 1$

b. if the gradient term dominates the scalar contribution behaves as $p_\phi \simeq -\frac{\rho_\phi}{3} \rightarrow w_\phi \simeq -\frac{1}{3}$

c. if the potential term dominates the scalar contribution behaves as $p_\phi \simeq -\rho_\phi \;\rightarrow\; w_\phi \simeq -1$

The behaviour of a scalar field as an effective cosmological constant is therefore related to kinematic properties, rather than to a particular potential: this kinematic condition which favours inflation is called **slow-roll**. For a scalar field holds $-1 \leq w_\phi \leq 1$, therefore a smooth connection with a decelerated universe ($w_\phi \geq -\frac{1}{3}$) is possible.

It is then possible that in a region of space of a given size, the scalar field is dominated by its potential energy: a nearly accelerated expansion can take place and inflate this patch to the size of our observable universe.

$$\begin{aligned} H^2 = \frac{1}{3M_{\rm pl}^2}\left[\frac{\dot{\phi}^2}{2} + V(\phi)\right] \rightarrow H^2 &\simeq \frac{V(\phi)}{3M_{\rm pl}^2} \\ \ddot{\phi} + 3H\dot{\phi} + V' = 0 \rightarrow 3H\dot{\phi} &\simeq -V' \end{aligned}$$

The slow-roll regime can occur for any sufficiently flat potential: the inflaton can be on a local maximum and roll towards a true minimum or it can find at a value displaced sufficiently from the minimum of the potential.

Necessary conditions for the slow-roll are $\epsilon_V \ll 1$ and, where $\eta_V \ll 1$ the slow-roll parameters and are defined as

$$\epsilon_V = \frac{M_{\rm pl}^2 V_\phi^2}{2V^2} \qquad \eta_V = \frac{M_{\rm pl}^2 V_{\phi\phi}}{V}$$

For a simple potential as $V(\phi) = \frac{m^2}{2}\phi^2$ the inflationary trajectory during slow-roll in which is:

$$
\begin{aligned}
H(t) &\simeq H_i - \frac{m^2}{3}(t - t_i)\,, \\
\phi(t) &\simeq \phi_i - \sqrt{\frac{2}{3}} m M_{\rm pl}(t - t_i)\,, \\
a(t) &\simeq a_i \exp\left[\frac{3}{2m^2}\left(H_i^2 - H^2\right)\right]
\end{aligned}
$$

This is called a transient attractor, when this trajectory is approached the inflaton stays close to it for a long time .....

... however the global fixed point of the homogeneous phase space is in the origin, in the so-called coherent oscillation regime:

$$
H(t) \simeq \frac{2}{3t}\left[1 + \frac{\sin 2mt}{2mt} + \mathcal{O}\left(\frac{1}{t^2}\right)\right]
$$

$$
\phi(t) \simeq \frac{2\sqrt{2} M_{\rm pl} \sin mt}{\sqrt{3} mt} + \mathcal{O}\left(\frac{1}{t^2}\right)
$$

which is characterized by a matter dominated universe ($H(t) = \frac{2}{3t}$) with a modulation of decreasing oscillations. Inflation has a graceful exit into a decelerated universe (a matter dominated period for this quadratic potential) in which the inflaton can decay into relativistic fields/particles and lead to a subsequent radiation thermalized dominated universe.

Magnetic fields in cosmology

- In case of an homogeneous PMF, our Universe would be described by a Bianchi metric.
- Inhomogenous PMFs respect homogeneity and isotropy globally, i.e. they can be described within a Robertson-Walker metric.
- Nucleosynthesis impose the following constraint: 300 nG (same order of magnitude for homogeneous or inhomogeneous magnetic field)

Grasso & Rubinstein 1995

- A stochastic background of PMF undergo a significant dissipation before recombination due to the large viscosity of the baryon-photon fluid. This dissipation results in distortions of the CMB black-body spectrum (in form of a chemical potential and Compton y distortions) constrained by FIRAS. The resulting constraint is 30 nG for the mean sqquare amplitude.

Jedamzik, Katalinic & Olinto 2000

Causality argument

Caprini & Durrer 2003

- Magnetic fields generated within classical Maxwell equations with a comoving length L by a random process which is statistically homogeneous and isotropic.

$$\langle B_i(\mathbf{k}) B_j^*(\mathbf{k}') \rangle = \frac{(2\pi)^3}{2} \delta(\mathbf{k} - \mathbf{k}') \Big[(\delta_{ij} - \hat{k}_i \hat{k}_j) P_S(k) + i \epsilon_{ijl} \hat{k}_l P_A(k) \Big]$$

$$P_S(k) = A_S \left(\frac{k}{k_*} \right)^{n_S} \qquad P_A(k) = A_A \left(\frac{k}{k_*} \right)^{n_A}$$

$$n_S \geq 2 \qquad n_A \geq 3$$

- These spectral indices are valid for scales larger than the coherence length of the stochastic background.
- If the generation is not during inflation then L is limited by the Hubble radius at the that time.
- This argument does not apply to inflation since there is a mismatch between the Hubble radius and particle horizon which grows quasi exponentially (and could be proportional to the coherence length of the physical process of amplification of quantum fluctuations).

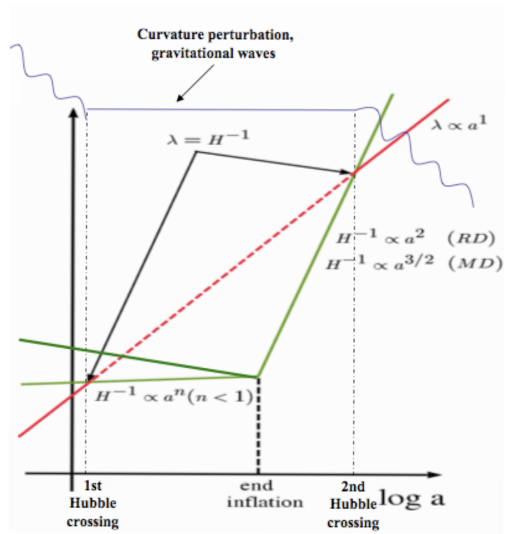
Primordial Magnetic Fields in the early Universe

- Post-inflationary mechanisms of generation of primordial magnetic fields suffer of a generically too small correlation length: @ QCD H^{-1} around O(1 pc), @ EWPT H^{-1} around O(1 au)

- Post-inflationary causal mechanism predict blue spectral indices

- Inverse cascade can help ....

- .... however a large correlation length for primordial magnetic fields is not a problem within the inflationary scenario.



- Concerning the spectrum, a scale-invariant spectrum of curvature perturbations can easily be generated:

$$k^3 |\mathcal{R}(k)|^2 \propto k^{n_s - 1}$$

$n_s = 0.960 \pm 0.007$ *(68%; Planck*+WP)

$$Y(\mathbf{k}, \tau) = \int d\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, Y(\mathbf{x}, \tau)$$

Amplification of quantum fluctuations by the expanding geometry

- A closer look to density perturbations which originate from the inflaton fluctuations. A (gauge-invariant) inflaton fluctuation satisfy the following equation of motion (prime denote derivative with respect to the conformal time $d\tau = dt/a$):

$$(a\delta\phi_k)'' + \left(k^2 - \frac{z''}{z}\right)(a\delta\phi_k) = 0 \qquad z = a\dot{\phi}/H \qquad \mathcal{R} = -\frac{a\delta\phi}{z}$$

- This applies to gravity waves as well

$$(ah_k^{+,\times})'' + \left(k^2 - \frac{a''}{a}\right)(ah_k^{+,\times}) = 0$$

- Exponential expansion in cosmic time corresponds in conformal time to: $a(\tau) = -\frac{1}{H\tau}\,, \quad \tau < 0$

- Vacuum quantum fluctuations when $k^2 >> z''/z\,, a''/a$

- Squeezed quantum states resembling classical states when $k^2 << z''/z\,, a''/a$

- These asymptotic behaviour can be obtained by exact solutions in terms of Hankel functions for a de Sitter evolution

- Gauge fields:

Lagrangian: $$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$ $$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

Eq. : $$\nabla^\mu F_{\mu\nu} = 0$$

$$\nabla_{[\gamma} F_{\mu\nu]} = 0$$ $$\mathbf{B} = \nabla \times \mathbf{A}$$

$$A''_{i\,k} + k^2 A_{i\,k} = 0$$ $$A_0 = 0\,, \quad \nabla \cdot \mathbf{A} = 0 \qquad (\nabla^\mu A_\mu = 0)$$

Coulomb gauge Lorenz gauge

- Why primordial density perturbations or gravitational waves differ from gauge fields?

Conformal invariance

- Properties of transformations with respect to a change in metric and fields

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$
$$\tilde{F}_{\mu\nu} = \Omega^s F_{\mu\nu}$$

think about g as the Minkowski metric in the following, the conformal weight as the scale factor

- Eqs of motion change in n-dimensions according to:

$$\tilde{\nabla}^\mu(\Omega^s F_{\mu\nu}) = \Omega^{s-2}\nabla^\mu F_{\mu\nu} + (n-4+s)\Omega^{s-3} g^{\mu\nu} F_{\mu\delta}\nabla_\nu \Omega$$

$$\tilde{\nabla}_{[\gamma}(\Omega^s F_{\mu\nu]}) = \Omega^s \nabla_{[\gamma} F_{\mu\nu]} + s\Omega^{s-1}(\nabla_{[\gamma}\Omega) F_{\mu\nu]}$$

Eqs of motion are invariant for $s = 0\,, \quad n = 4$

- If F has conformal weight equal to zero, the gauge potential has the same

$$\tilde{F}_{\mu\nu} = \tilde{\nabla}_\mu A_\nu - \tilde{\nabla}_\mu A_\nu = \nabla_\mu A_\nu - \nabla_\mu A_\nu = F_{\mu\nu}$$

- Analogously for minimally coupled scalar field:

$$\tilde{\phi} = \Omega^s \phi$$

- Eqs of motion change in n-dimensions according to:

$$\tilde{g}^{\mu\nu}\nabla_\mu\nabla_\nu(\Omega^s\phi) = \Omega^{s-2}g^{\mu\nu}\nabla_\mu\nabla_\nu\phi + (2s+n-2)\Omega^{s-3}g^{\mu\nu}(\nabla_\mu\Omega)(\nabla_\nu\phi)$$
$$+s\,\Omega^{s-3}\,\phi\,g^{\mu\nu}\nabla_\mu\nabla_\nu\Omega + s(n+s-3)\Omega^{s-4}\,\phi\,g^{\mu\nu}\nabla_\mu\Omega\nabla_\nu\Omega$$

Conclusions: quantum fluctuations of minimally coupled scalar fields are amplified by the expanding space-time, as gravitons, gauge fields are not.

Large scale magnetic fields in cosmology are not almost for free as density perturbations/ gravitons apparently ....

A (non-exhaustive) list

- Magnetic fields from cosmological perturbations
- Breaking gauge invariance U(1) through gravitational couplings
- Quantum effects in curved space-time (vacuum polarization or anomaly)
- Coupling to a charged scalar field (scalar QED)
- Coupling to a pseudo-scalar field
- F^2 coupling to a scalar field (Inflation, Pre Big-Bang scenario, Anisotropic inflation)
- (p) Reheating

Magnetic fields from vorticity?

- Magnetic fields generated electric currents induced by the difference in motion of protons and electrons.
- Vorticity of the plasma can produce such electric currents. Harrison 1970
- Since linear vector perturbations decay in a perfect fluid dominated Universe, need to study cosmological perturbations at second order keeping separated electron, protons and photons with Boltzmann equation.
- Takahashi et al. (2005) concluded that it is possible to obtain B of order (10^{-25} G) on comoving scales of 10 Mpc.

Notes for inflationary mechanisms

- Gauge field not necessarily U(1) of the standard model, but could be hypermagnetic fields, which convert into standard magnetic fields with O(1) coefficients.

- Electric fields dumped by the large conductivity of the primeval plasma once charged particles are generated during (p)reheating

- Magnetic fields remain frozen in by the large conductivity: $B\, a^2$ = constant during the radiation stage

- Presentation based on the pioneering papers. Be aware that quantitative predictions are changed since the time these papers have been written: the astrophysical understanding and the existing constraints on PMF have changed substantially since 1980 or 1990.

Breaking gauge-invariance

- Breaking gauge invariance U(1) through gravitational couplings

Turner & Widrow 1988

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{b}{2}RA_\mu A^\mu - \frac{c}{2}R_{\mu\nu}A^\mu A^\nu$$

- F_k Fourier transform of a^2 B

$$F_k'' + k^2 F_k + QF_k = 0 \qquad Q = 6b\frac{a''}{a} + c\left(\frac{a''}{a} + \frac{a'^2}{a^2}\right) = \frac{n}{\eta^2}$$

- Tune b,c during de Sitter era, but also during radiation and matter.
- For n < 0 you obtain super-adiabatic amplification. Be aware that for n > 0 you have to take into account suppression of fluctuations during inflation.
- Need to take into account back-reaction from magnetic fields on the inflaton.

Demozzi, Mukhanov & Rubinstein 2009

Quantum effects

- One loop vacuum polarization in curved space-time

Turner & Widrow 1988

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_g$$

$$\mathcal{L}_g = -\frac{1}{4m_e^2}\left[bRF_{\mu\nu}F^{\mu\nu} + cR_{\mu\nu}F^{\mu\kappa}F^{\nu}_{\kappa} + dR_{\mu\nu\lambda\kappa}F^{\mu\nu}F^{\lambda\kappa}\right]$$

$$A_i'' + sA_i' \simeq 0\,, \quad k\eta << 1\,, \quad s = -6\frac{1+w}{1+3w}$$

- Quantum conformal anomaly (from other fields)

Dolgov 1993

$$\nabla_\mu F^\mu_\nu + \kappa\frac{\partial_\mu a}{a}F^\mu_\nu = 0$$

$$A_i'' + \kappa\mathcal{H}A_i' + k^2 A_i = 0$$

SU(N) gauge theory in presence of N_f charged fermions

$$\kappa = \alpha\pi\left(\frac{11}{3}N - \frac{2}{3}N_f\right)$$

$\kappa \sim 0.06 \quad \text{for} \quad SU(5) \quad \text{with} \quad N_f = 3$

$\kappa \sim \mathcal{O}(1)$ to have interesting effects

Scalar QED

Turner & Widrow 1988
Calzetta, Kandus, Mazzitelli 1998
Giovannini & Shaposhnikov 2000

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - (D_\mu \phi)^* (D^\mu \phi) - V(|\phi|) \qquad D_\mu = \nabla_\mu - ieA_\mu$$

$$\nabla_\mu F^\mu_\nu = -j_\mu + 2e^2 A_\mu |\phi|^2 \qquad j_\mu = ie(\phi \partial_\mu \phi^* - \phi^* \partial_\mu \phi)$$

- Coulomb gauge: $\nabla \cdot A = 0$
- Charged scalar field: $\phi = \frac{\rho}{\sqrt{2}} e^{i\theta}$

$$A''_{T\,k} + (k^2 + e^2 a^2 \rho^2) A_{T\,k} = a^2 \delta j_{T\,k}$$

Two effects to take into account:

a. effective mass which breaks conformal invariance during inflation and generates a blue spectrum for the gauge fields

b. creation of currents since the complex scalar field is generated during inflation

Giovannini & Shaposhnikov 2000 concluded that $B_{dec}/T^2_{dec} = O(10^{-40})$

Pseudo-scalar field

Garretson, Field & Carroll 1992
Finelli & Gruppuso 2000
Anber & Sorbo 2006

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi) - \frac{g}{4}\phi F_{\mu\nu}\tilde{F}^{\mu\nu} \qquad \tilde{F}^{\mu\nu} = \frac{\epsilon^{\mu\nu\alpha\beta}}{2}F_{\alpha\beta}$$

Convenience in studying left and right circular polarization:

$$A''_{\pm\, k} + (k^2 \pm gk\phi')A_{\pm\, k} = 0 \qquad \phi' \simeq -\frac{\sqrt{2\epsilon}M_{\rm pl}}{\tau}$$

With the above time dependence for the variation of the scalar field the eq:

$$A''_{\pm\, k} + (k^2 \pm 2k\frac{\xi}{\tau})A_{\pm\, k} = 0 \qquad \xi \equiv g\,\mathrm{M_{pl}}\sqrt{\frac{\epsilon}{2}}$$

admits as exact solution those of the Coulomb wave equation with zero angular momentum (L=0):

$$\frac{d^2F}{d\rho^2} + \left[1 - \frac{2\xi}{\rho} - \frac{L(L+1)}{\rho^2}\right]F = 0$$

A stochastic background of PMF is generated with maximal helicity (the + component is copiously generated).

The choice for positive frequency at early time selects the particular form of the solution:

$$A_{+\,k}(\tau) = \sqrt{\frac{1}{2k}}\,[i\,F_0(\xi,\,-k\tau) + G_0(\xi,\,-k\tau)]$$

where F_0 and G_0 are the regular and irregular Coulomb wave functions, respectively.

On large scales:

$$A_{+\,k}(\tau) \simeq \sqrt{\frac{1}{2k}}\left(\frac{k}{2\xi\,aH}\right)^{1/4} e^{-2\sqrt{2\xi\,k/aH}+\pi\,\xi}\,, \quad k|\tau| \ll 2\xi$$

Blue spectrum generated around Hubble crossing.

The final magnetic field generated is estimated as:

$$B \simeq 10^{-113/4}\frac{e^{\pi\xi}}{\xi^{17/12}}\left(\frac{T_{\rm RH}}{10^9{\rm GeV}}\right)^{11/36}\left(\frac{l}{1\,{\rm Mpc}}\right)^{-9/4}{\rm nG}$$

Anber & Sorbo 2006

$\xi \lesssim 5$ to avoid strong back-reaction on the inflaton

F^2 coupling

$$\mathcal{L} = -\frac{f^2(\phi)}{4} F_{\mu\nu} F^{\mu\nu} - \frac{g^{\mu\nu}}{2} \partial_\mu \phi \partial_\nu \phi - V(\phi)$$

$$A''_{T\,k} + 2\frac{f'}{f} A'_{T\,k} + k^2 A_{T\,k} = 0$$

- String gravity, dilaton — Ratra 1992
- Pre-Big Bang scenario — Gasperini, Giovannini & Veneziano
- Supergravity — Martin & Yokoyama 2008
- Anisotropic inflation — Watanabe, Kanno & Soda 2008

Power-law inflation

Ratra 1992

$$f(\phi) \propto e^{\alpha\phi/(2M_{\rm pl})} \qquad V(\phi) \propto e^{-\lambda\phi/\mathrm{M}_{\rm pl}}$$

The large scale magnetic field studied during power-law inflation driven the scalar fields is predicted as:

$$|B(k)|^2 \propto k^{2-2|\nu|} \qquad \nu = \frac{1}{2} + \alpha\frac{\lambda}{1-\lambda^2}$$

Amplitude of the order of nG can be obtained.

In the subsequent radiation dominated stage:

$$A''_{T\,k} + 4\pi a^2 \sigma A'_{T\,k} + k^2 A_{T\,k} = 0$$

where σ is the conductivity.

Ratra is also the first one to argue that the electric field is exponentially damped due to the conductivity. By assuming an instantaneous change at t_R in which the inflaton freezes and the conductivity jumps to a finite value:

$$E_i^{\rm rad}(t_R, \mathbf{x}) = e^{-4\pi\sigma t_R} E_i^{\rm inf}(t_R, \mathbf{x})$$

The magnetic field is unchanged at the transition.

String Cosmology

Gasperini, Giovannini, Veneziano

String effective action obtained after reduction from ten to four dimensions:

$$S = -\int d^4x\sqrt{-g}e^{-\phi}\left(R + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) + \ldots$$

$$A''_{T\,k} + \left(k^2 + e^{\phi/2}(e^{-\phi/2})''\right)A_{T\,k} = 0$$

Differently from the previous inflationary scenario, there are two different regimes before the radiation stage in which the dilaton is fixed to a constant value:

- the dilaton phase driven by the kinetic energy of the dilaton (super-inflation with $\dot{H} > 0$)

- the stringy intermediate phase with constant $\dot{\phi}$ preceding the graceful exit in a standard radiation with the dilaton frozen

PMF on Mpc scales can be naturally generated tuning the parameters of the model.

Anisotropic Inflation

The coupling f[2] can support also anisotropic inflation

Watanabe, Kanno & Soda 2008

Anisotropic extension of RW metric:

$$ds^2 = -dt^2 + a^2(t)\left[e^{-4\sigma(t)}dx^2 + e^{2\sigma(t)}(dy^2 + dz^2)\right]$$

$$A_x(t) \neq 0$$

$$\dot{A}_x = f^{-2}e^{-4\sigma}p_A/a$$

Relevant equations include:

$$\ddot{\sigma} + 3H\dot{\sigma} = \frac{e^{-4\sigma(t)}}{3M_{\rm pl}^2 f^2 a^4}$$

The relation $f \propto \frac{1}{a^2} \propto e^{\int d\phi \frac{2V}{M_{\rm pl}^2 V_\phi}}$ is the threshold case in which the shear contribution to the Einstein equation is nearly constant in time.

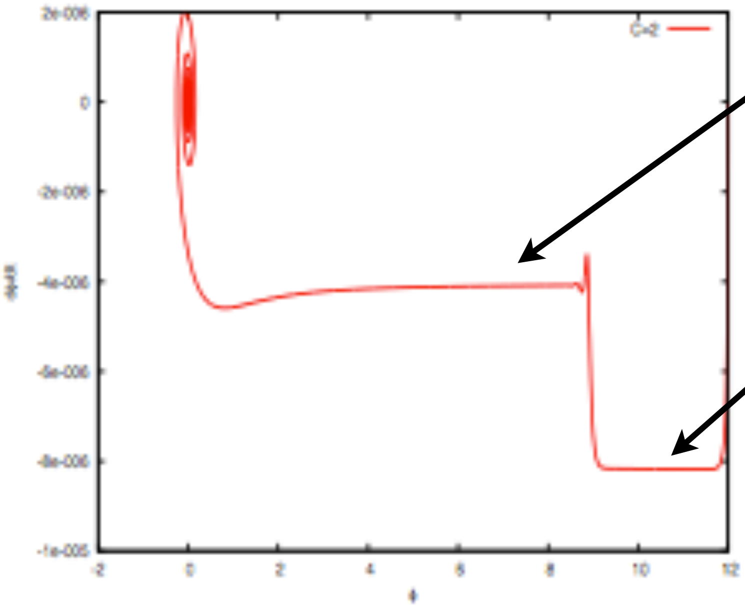
In general, if

$$f = e^{2c\int d\phi \frac{V}{M_{\rm pl}^2 V_\phi}}$$

when c>1 the vector contribution cannot be neglected. This situation is an attractor in phase space.

Example:

$$f(\phi) = e^{c\phi^2/(2M_{\rm pl}^2)} \qquad V(\phi) = \frac{m^2}{2}\phi^2$$



Gauge fields contribute to slow-roll inflation

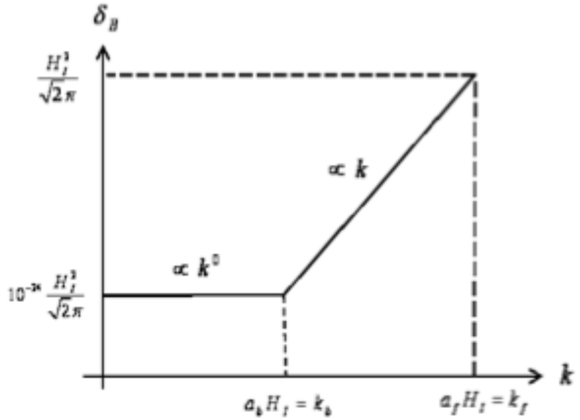
Gauge fields are negligible, standard slow-roll

inflation driven by the scalar fields

$$\frac{\dot\sigma}{H} \simeq \frac{c-1}{3c}\epsilon$$

For this choice of f the spectrum of gauge-field fluctuations can be quite steep in the infrared. In particular, the primordial spectrum for B would have a scale invariant spectrum for c=3/2 with an amplitude at present of pG assuming an energy scale during inflation of the order $10^{-6}\, M_{\rm pl}$.

However, this is a model in which back-reaction can be self-consistently taken into account, in particular for the value of c=3/2 corresponding to a scale-invariaant spectrum for the PMF.



Kanno, Soda & Watanabe 2009

Taking into account back-reaction a scale-invariant spectrum of PMF could be generated only on large scales with a suppression factor of 10^{-24} !

This particular model shows the importance of back-reaction when relevant.

Anisotropic inflation induce a direction dependence in th espectrum of primordial perturbations, called quadrupolar modulation, very interesting in the context of violation of isotropy on large angular scales confirmed by Planck 2013. Unfortunately this class of models leads to a quadrupolar modulation which is strongly constrained by WMAP (Hanson, Lewis & Challinor 2009) and Planck data (Kim & Komatsu 2013).

(p)Reheating

Finelli & Gruppuso 2000
Bassett, Pollifrone, Tsujikawa & Viniegra 2001

For the models in which the conformal invariance of gauge fields is broken because of the coupling to the inflaton, the coherent oscillation stage after slow-roll could provide an additional stage in which primordial magnetic fields can be further amplified.

The results are highly model dependent, since the non-perturbative decay into gauge fields depends strongly on the inflaton potential.

Example for scalar QED: $V(\phi) = \lambda(\phi^* \phi)^2$

The universe expands as radiation dominated when averaged through the coherent oscillations of the scalar field.

The homogeneous equation for the transverse component of the gauge field is

$$A''_{T\,k} + \left(k^2 + e^2 a^2 \rho^2\right) A_{T\,k} = 0$$

has a time dependent mass, which is described by the square of an elliptic cosine and is called Lame' equation (similar to the more famous Mathieu equation)

The solution exhibits exponential growth in the resonance bands:

$$A_{T\,k} \sim e^{\mu_k \tau}$$

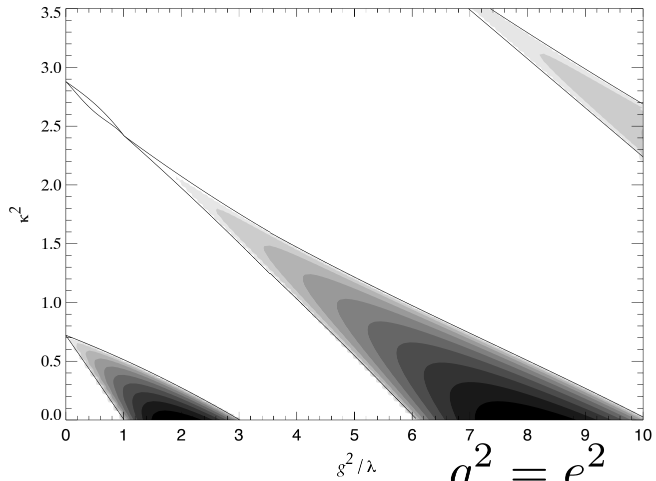
Low momenta are amplified when

$$n(2n-1) < e^2/\lambda < n(2n+1)$$

with n as an integer. The maximum value for the Floquet index is $\mu_k \simeq 0.238$

For n=1 gauge field fluctuations are not significantly dumped during inflation and therefore the net gain is really exponential.

$$\kappa^2 = \frac{k^2}{\lambda \phi^2_{\rm coh}}$$



$g^2 = e^2$

Greene, Kofman, Linde, Starobinsky 1997

Example for a pseudo-scalar field:

$$A''_{\pm\,k} + (k^2 \pm gk\phi')A_{\pm\,k} = 0$$

A typical potential is: $V(\phi) = V_0\,[1 + \cos(\phi/f)]$

Since the minimum of the potential can be approximated as quadratic, the coherent oscillations can be described as $\phi(t) \sim \cos(t/f)/t$ and therefore the oscillations slowly decrease with time, and the growth is not exponential.