Exoplanet Microlensing II:

Binary Orbits, Degeneracy & Calculation

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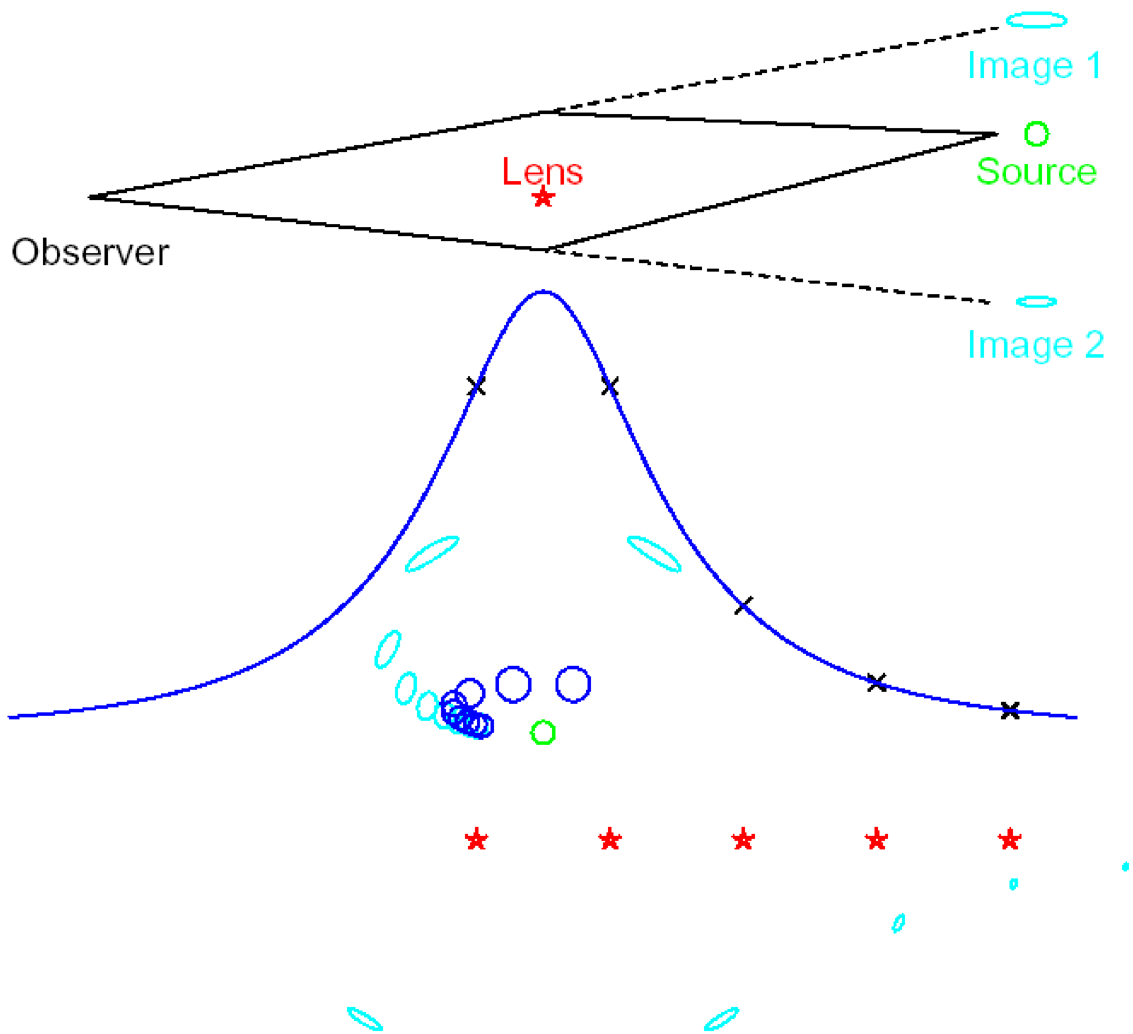


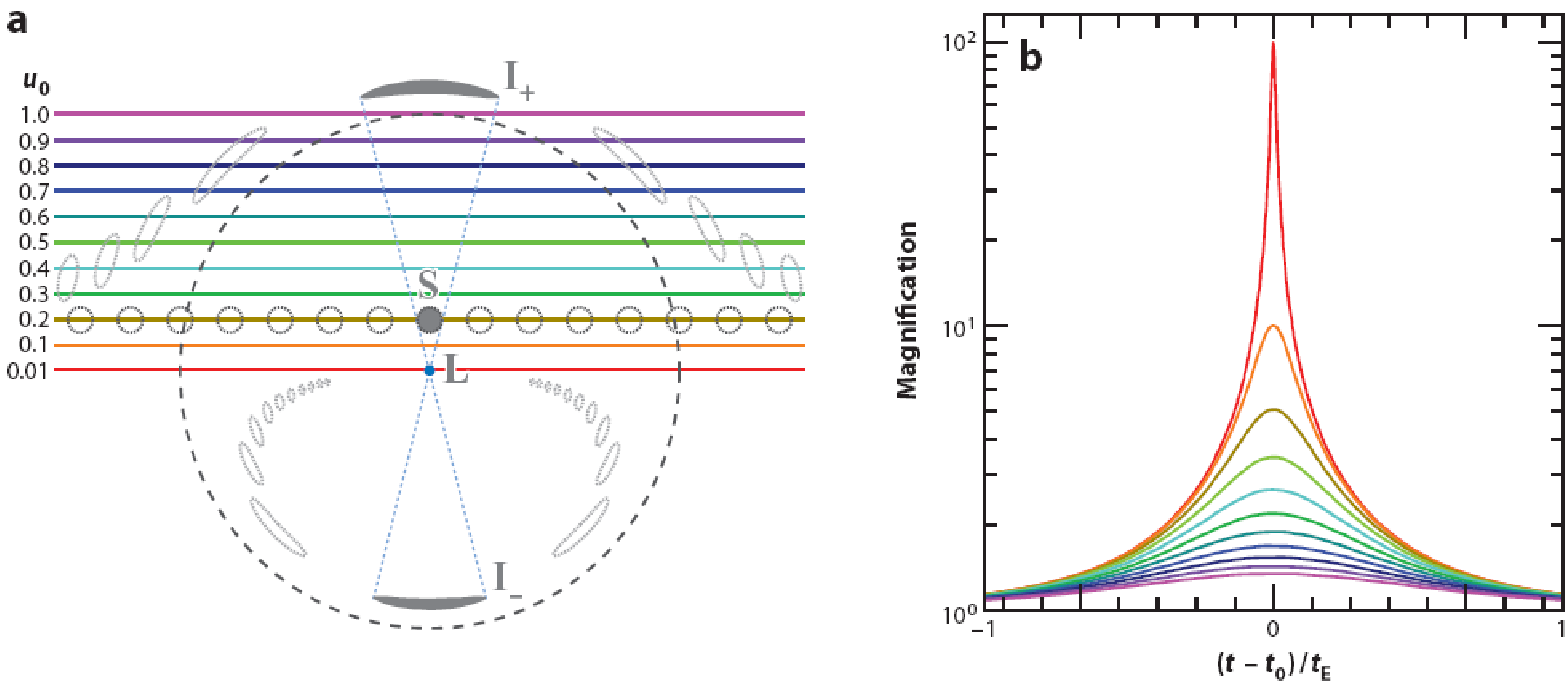
Simple Point Lens

3 Features & 3 Parameters

- Time of Peak
- Height of Peak
- Width of Peak

- t_0
- u_0
- t_E

Point-Lens Light Curves



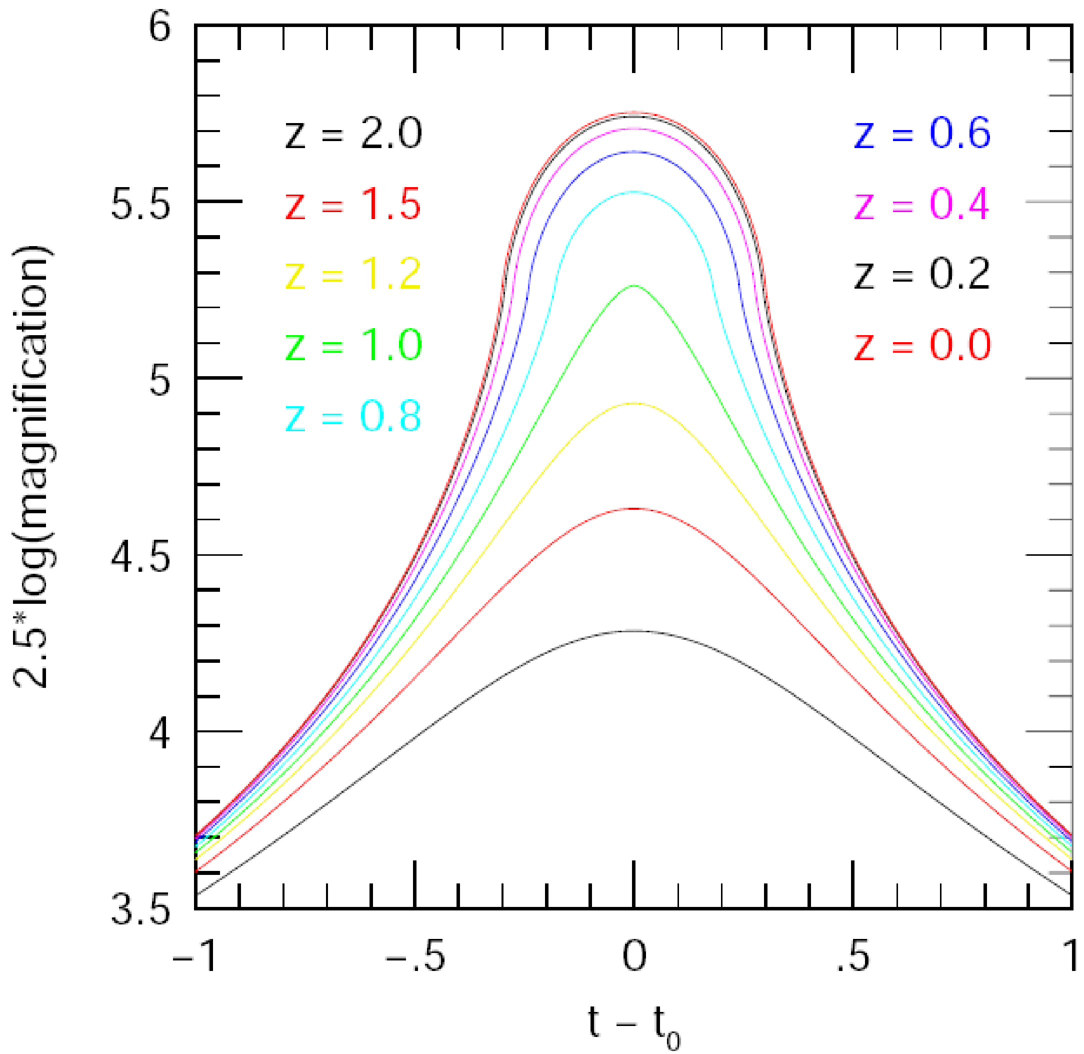
Point Lens + Finite Source Effect

4 Features & 4 Parameters

- Time of Peak
- Height of Peak
- Width of Peak
- Width of Cap

- t_0
- u_0
- t_E
- $t_* = \rho * t_E$

Finite Source “Attenuation”



Point Lens + Parallax

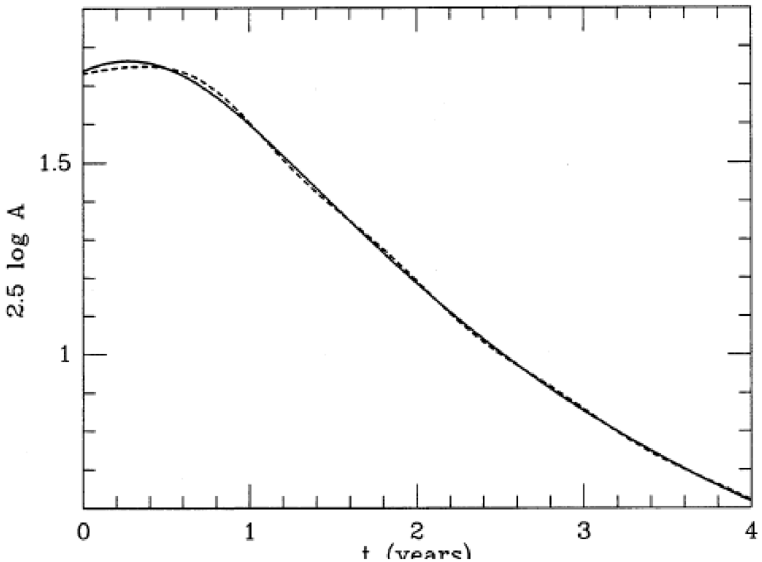
5 Features & 5 Parameters

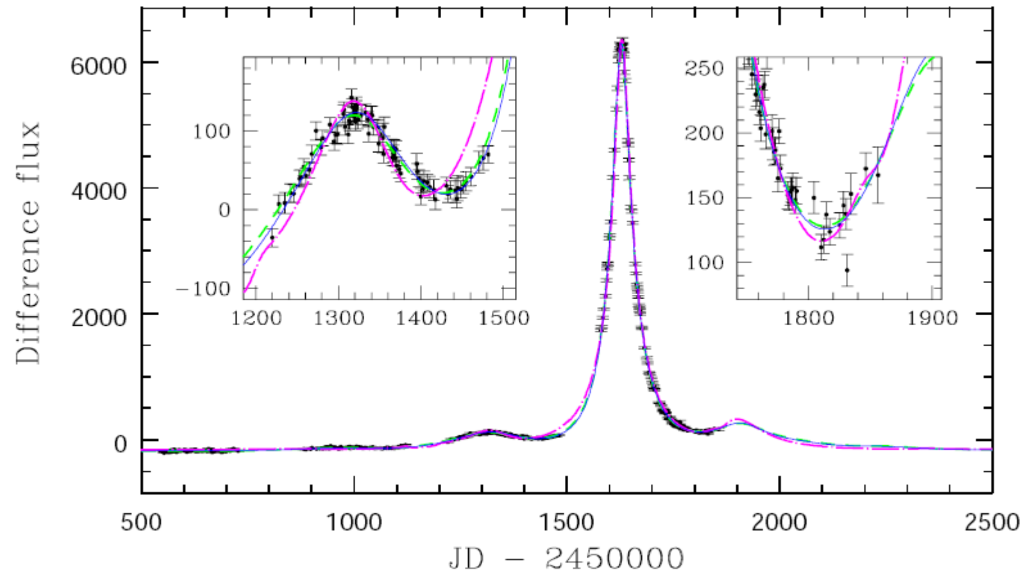
- Time of Peak
- Height of Peak
- Width of Peak
- Symmetric Distortion
- Anti-Symmetric Distortion

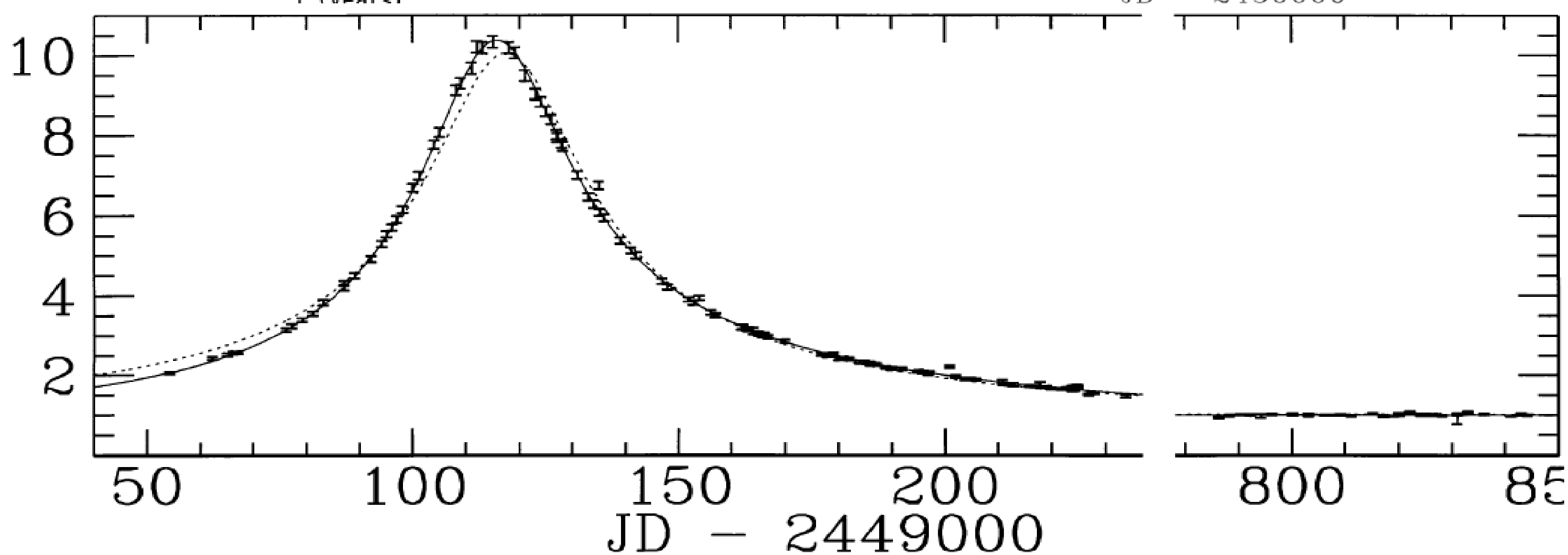
- t_0
- u_0
- t_E
- $\pi_{E,perp}$
- $\pi_{E,parallel}$

Parallax Examples

Many Year, Few Year, <1 Year







Point Lens + Parallax + FS

6 Features & 6 Parameters

- Time of Peak
- Height of Peak
- Width of Peak
- Width of Cap
- Symmetric Distortion
- Anti-Symmetric Distortion

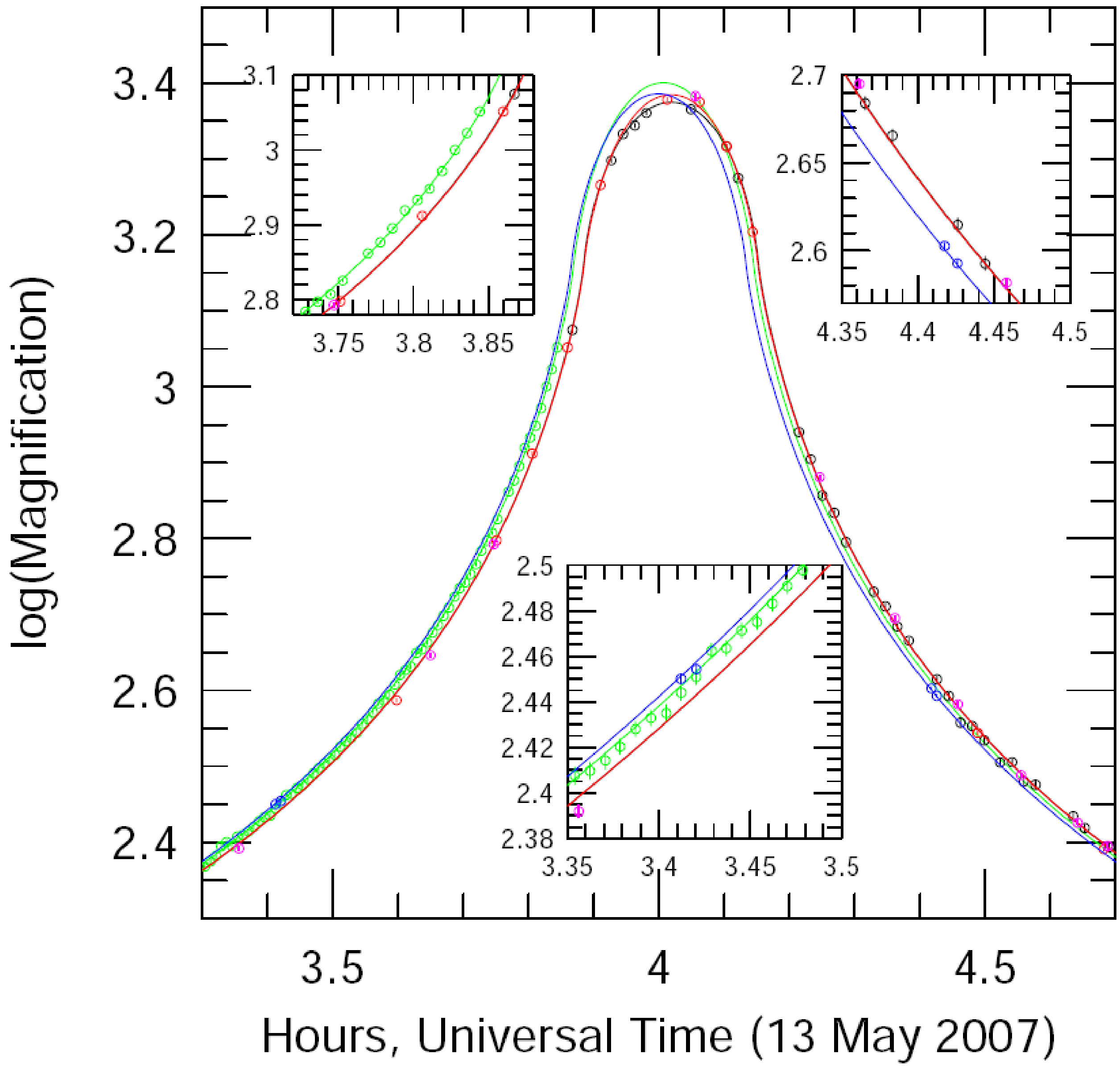
- t_0
- u_0
- t_E
- $t_* = \rho * t_E$
- $\pi_{E,perp}$
- $\pi_{E,parallel}$

Real Examples:

NONE

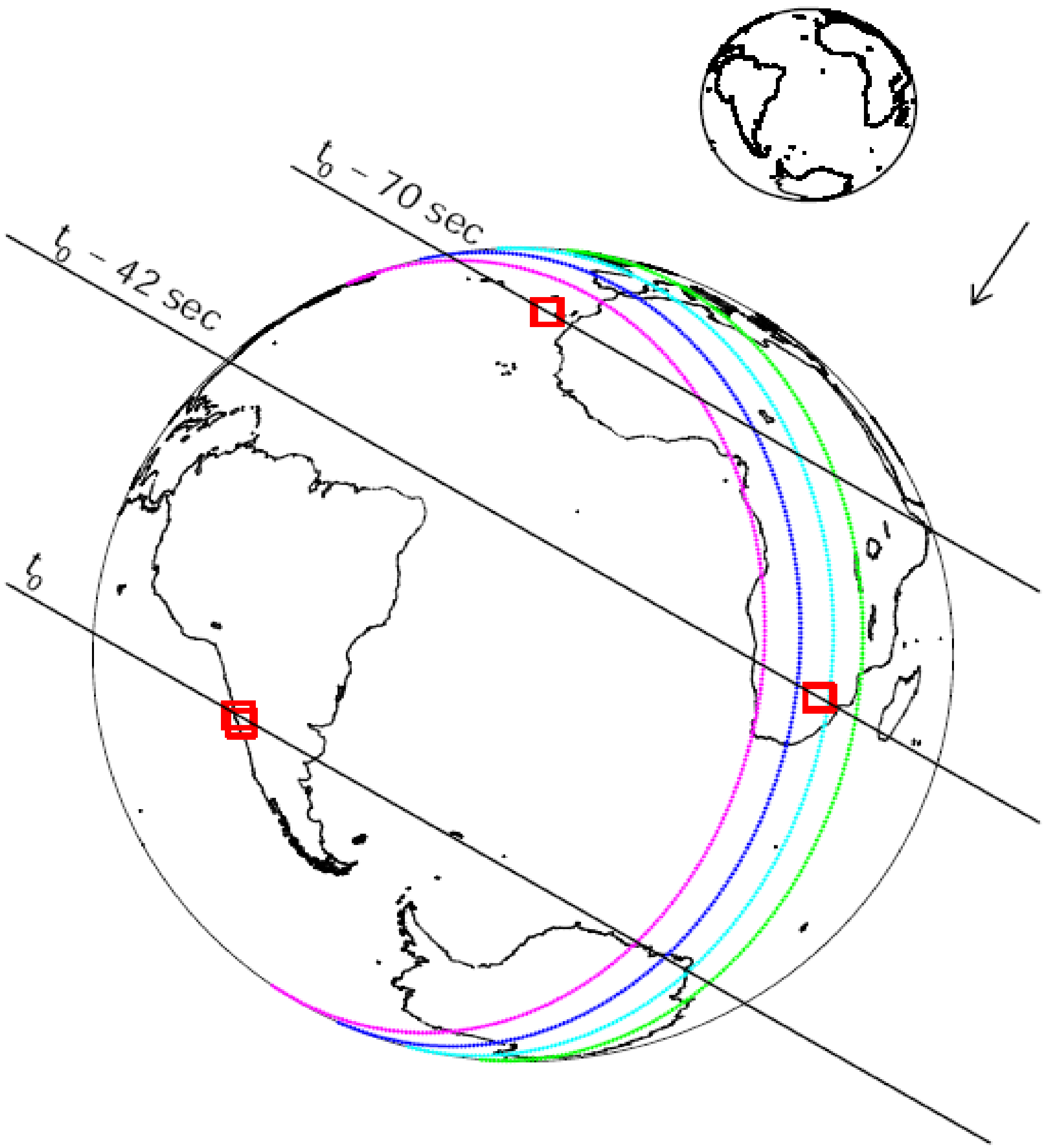
OGLE-2007-BLG-224

Canaries South Africa Chile



Terrestrial Parallax:

Simultaneous Observations on Earth



Simple Planetary (G&L) Lenses

6 Features & 6 Parameters

- Time of Peak
- Height of Peak
- Width of Peak
- Time of Perturbation
- Height of Perturbation
- Width of Perturbation

- t_0
- u_0
- t_E
- Trajectory angle: α
- Planet-star separation: s
- Planet/star mass ratio: q

Planetary Lenses usually have FS

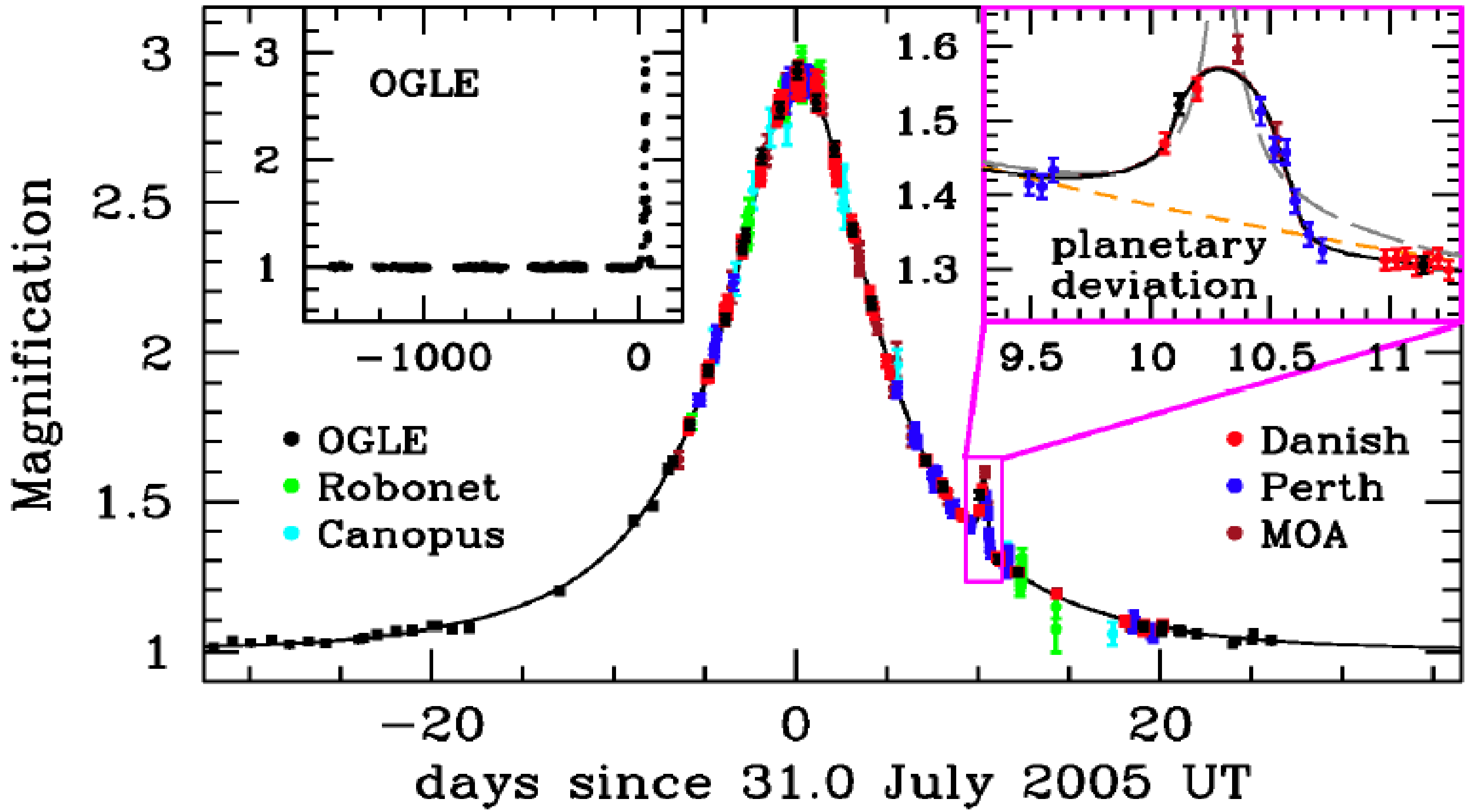
7 Features & 7 Parameters

- Time of Peak
- Height of Peak
- Width of Peak
- Time of Perturbation
- Height of Perturbation
- Width of Perturbation
- Width of Caustic Cr.

- t_0
- u_0
- t_E
- Trajectory angle: α
- Planet-star separation: s
- Planet/star mass ratio: q
- $t_* = \rho * t_E$

OGLE-2005-BLG-390

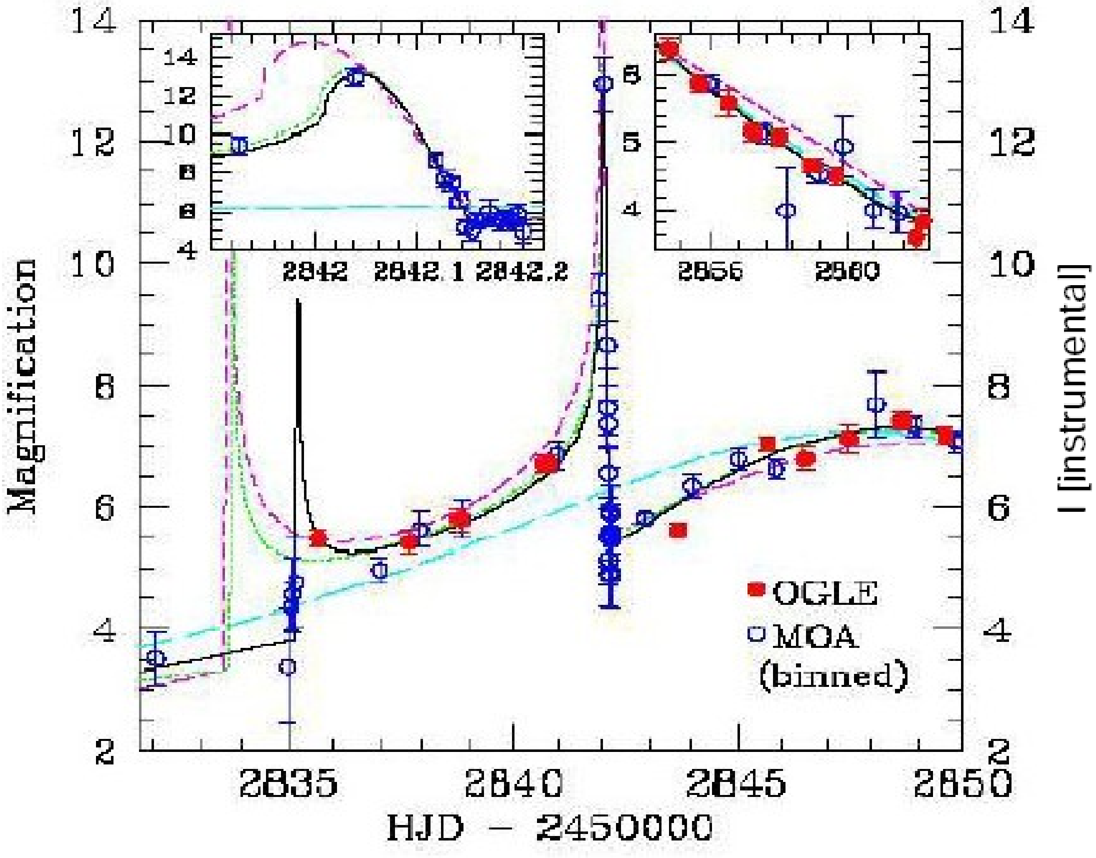
First Simple (G&L) Planetary Lens

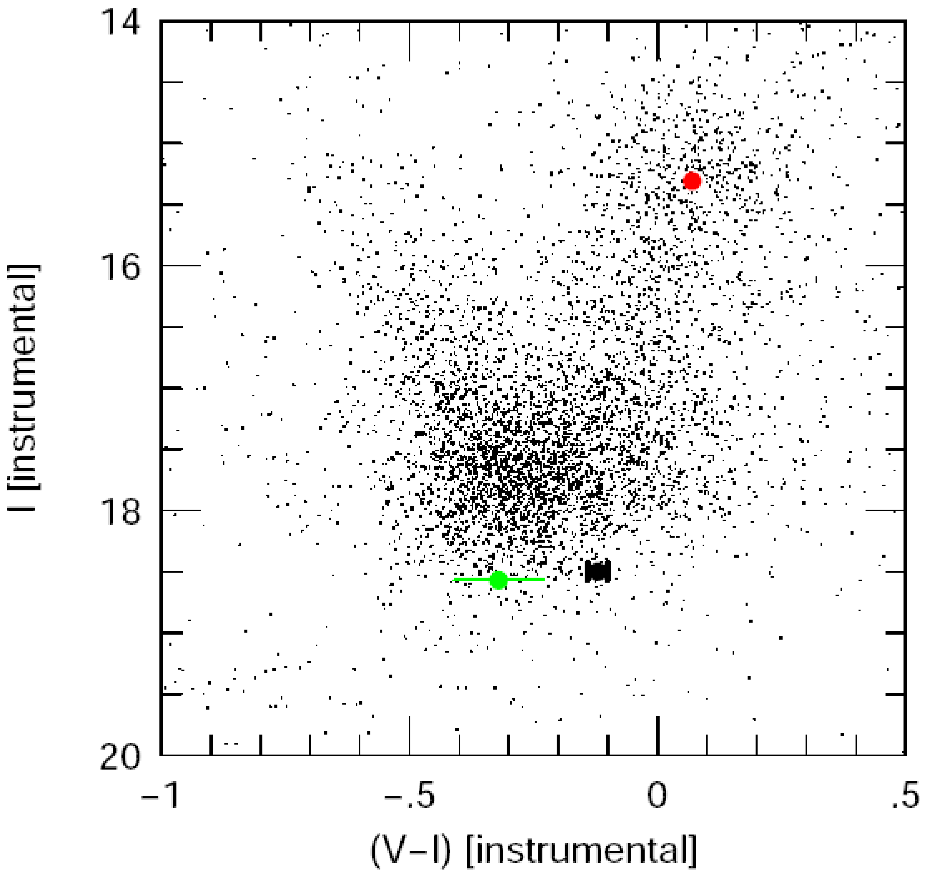


Beaulieu et al. 2006, Nature, 439, 437

First Microlensing Planet

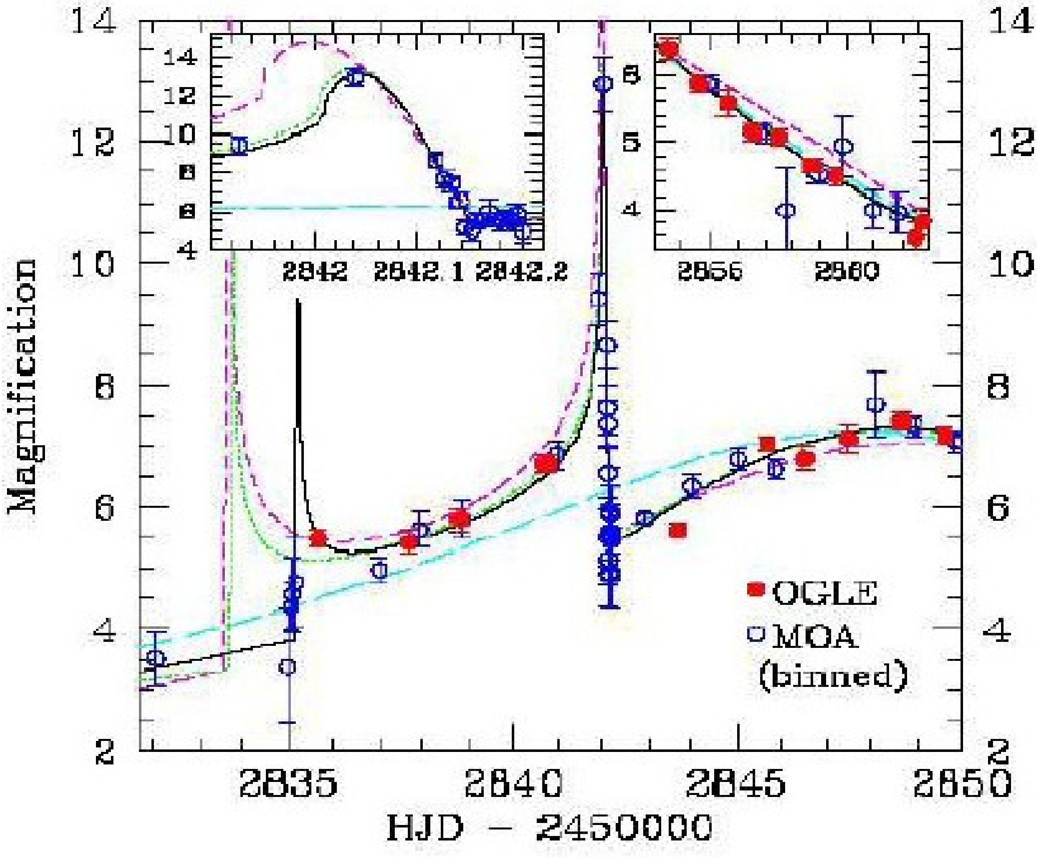
Pronounced Finite Source Effects

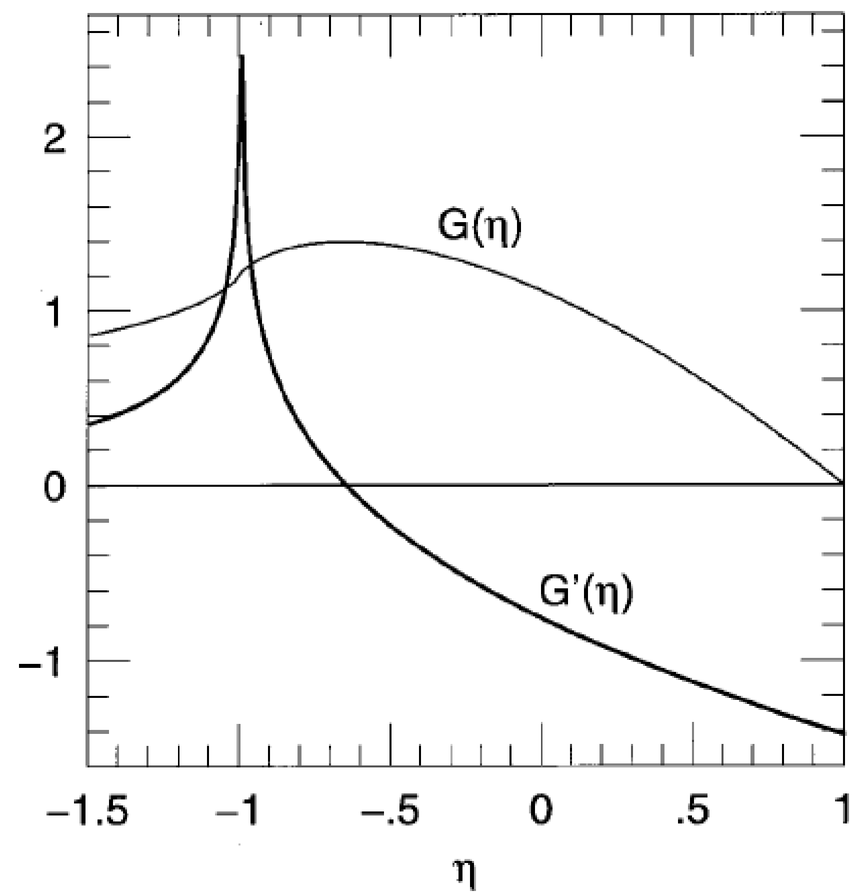




First Microlensing Planet

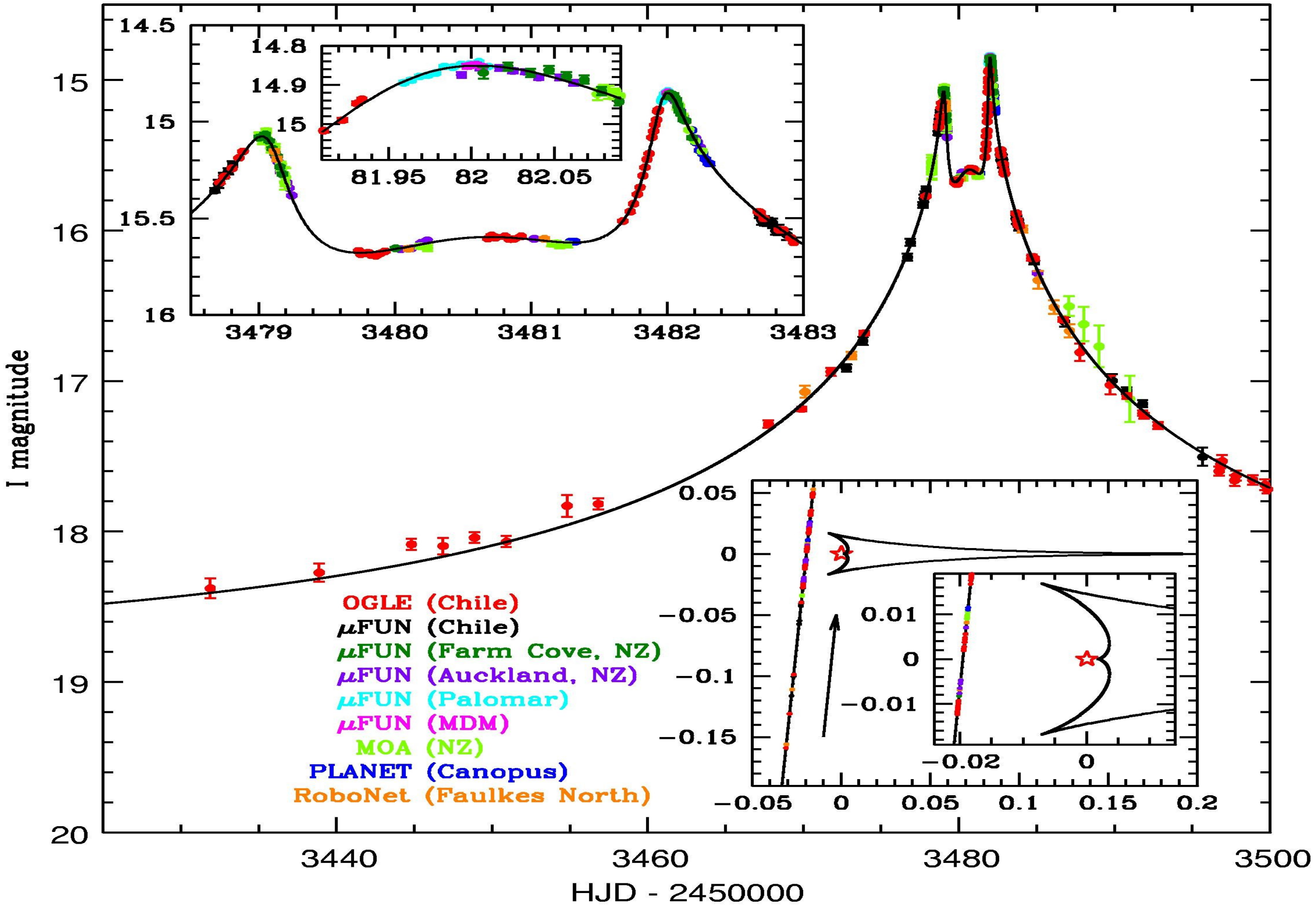
Perfect Fold Caustic Crossing





Second Microlensing Planet

Weak Finite Source Effects



Udalski et al. 2005, ApJ, 628, L109

Planet Lenses Often Have Parallax

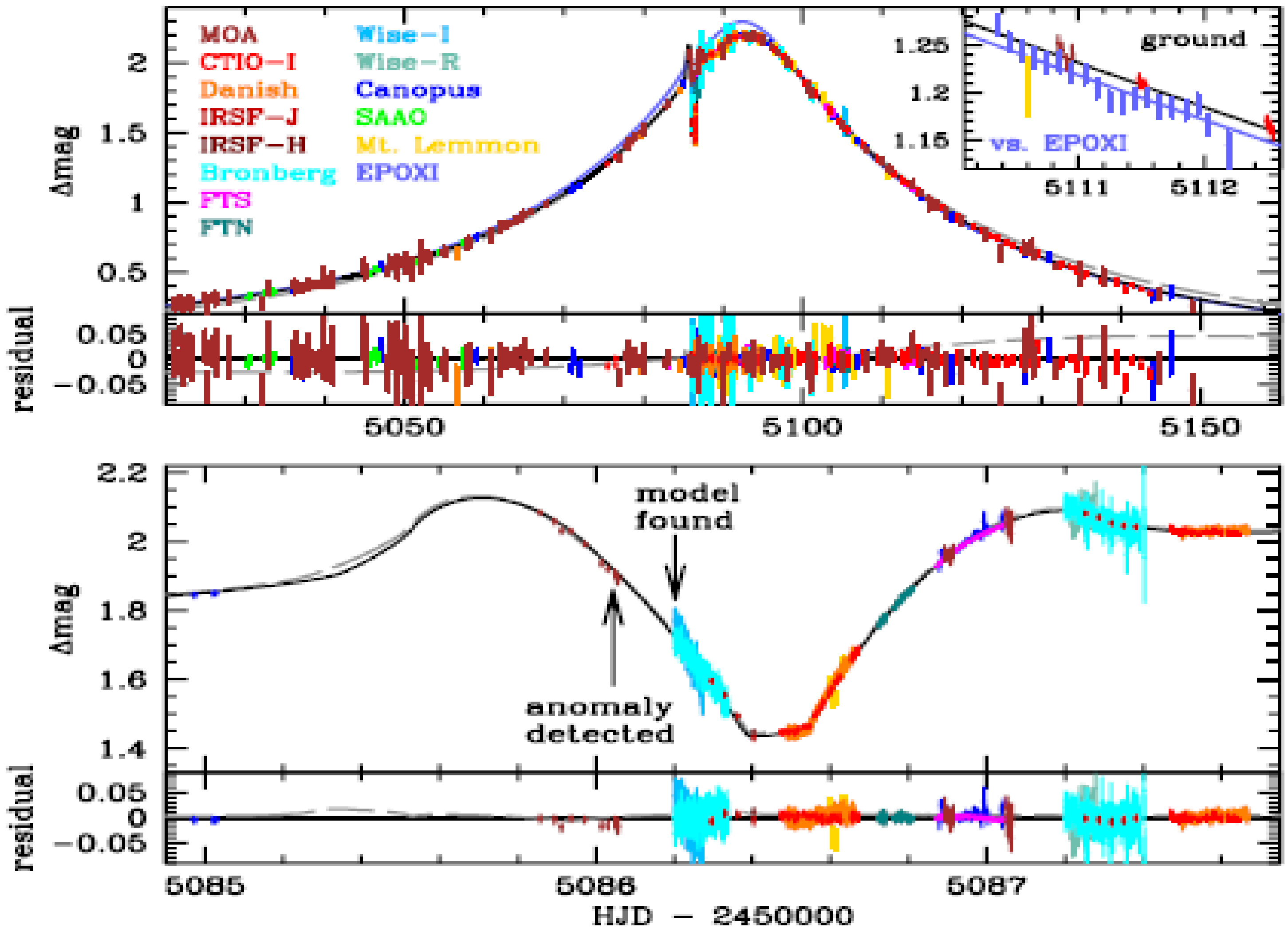
9 Features & 9 Parameters

- 3 Point-Lens
- Time of Perturbation
- Height of Perturbation
- Width of Perturbation
- Width of Caustic Cr.
- Symmetric Distortion
- Anti-symmetric Dist.

- t_0, u_0, t_E
- Trajectory angle: α
- Planet-star separation: s
- Planet/star mass ratio: q
- $t_* = \rho * t_E$
- $\pi_{E,perp}$
- $\pi_{E,parallel}$

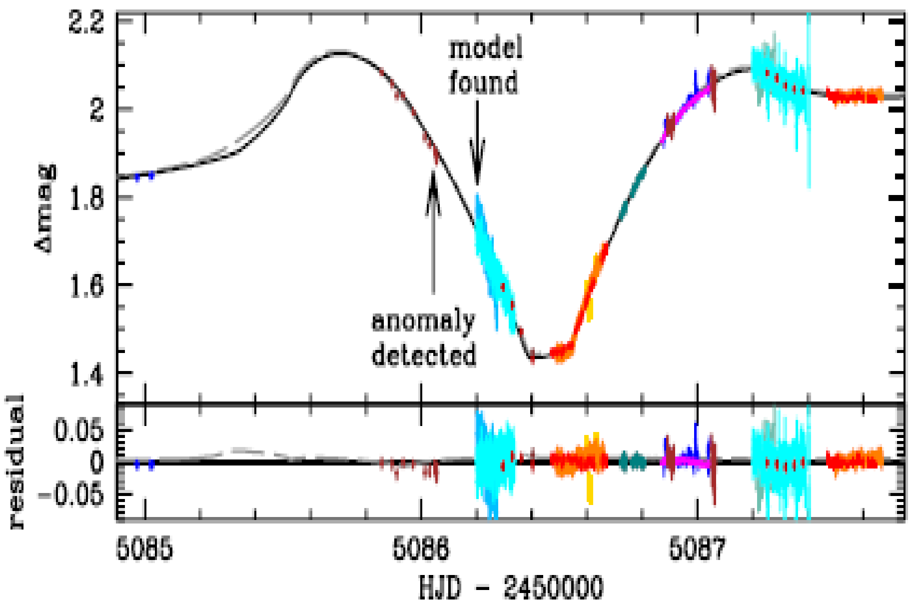
MOA-2009-BLG-266

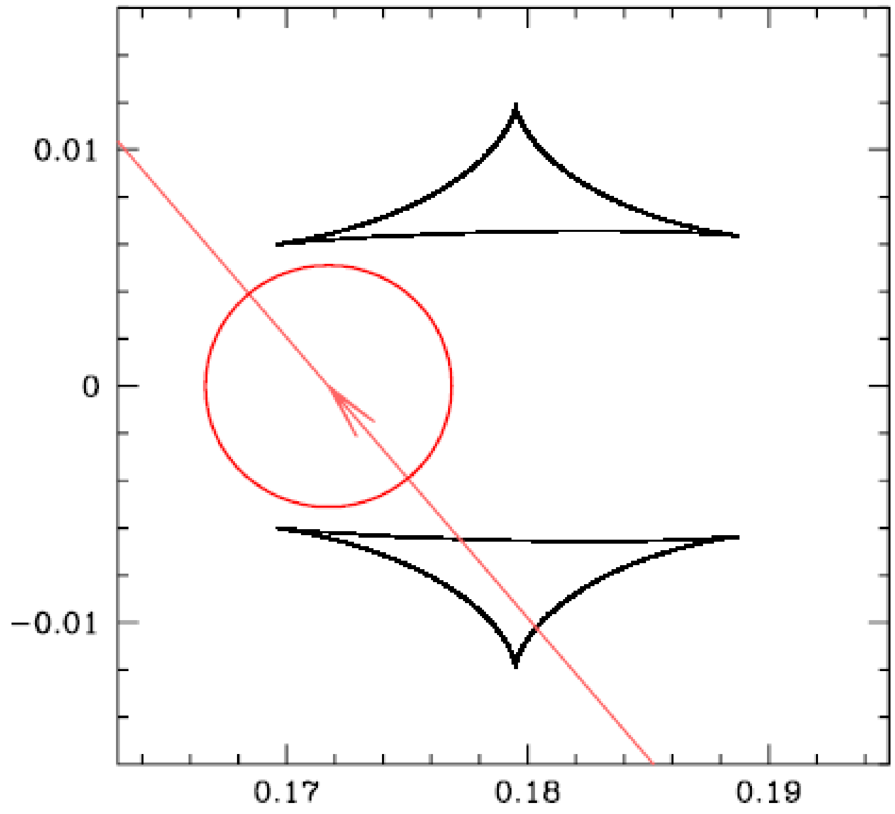
Parallax + Finite Source



Muraki et al. 2011, ApJ, 741, 22

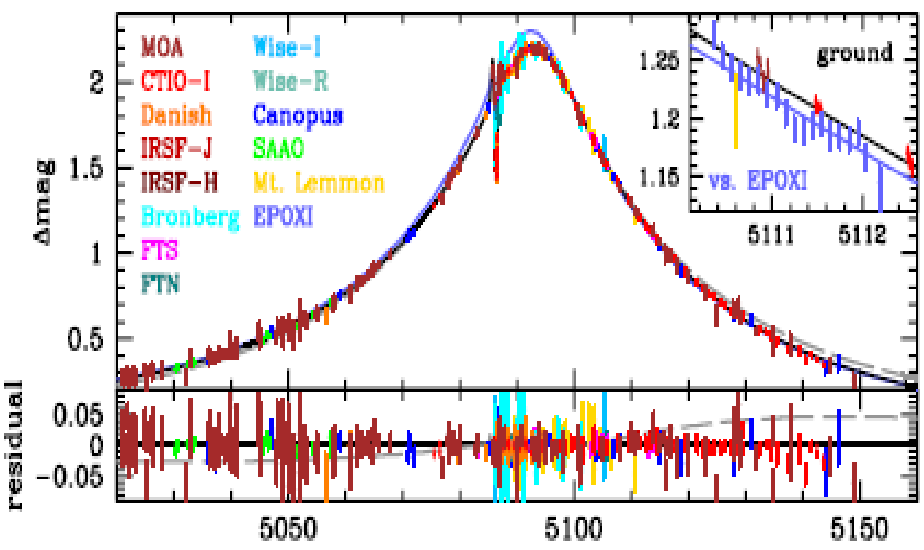
ρ Well-Measured from "Dip"

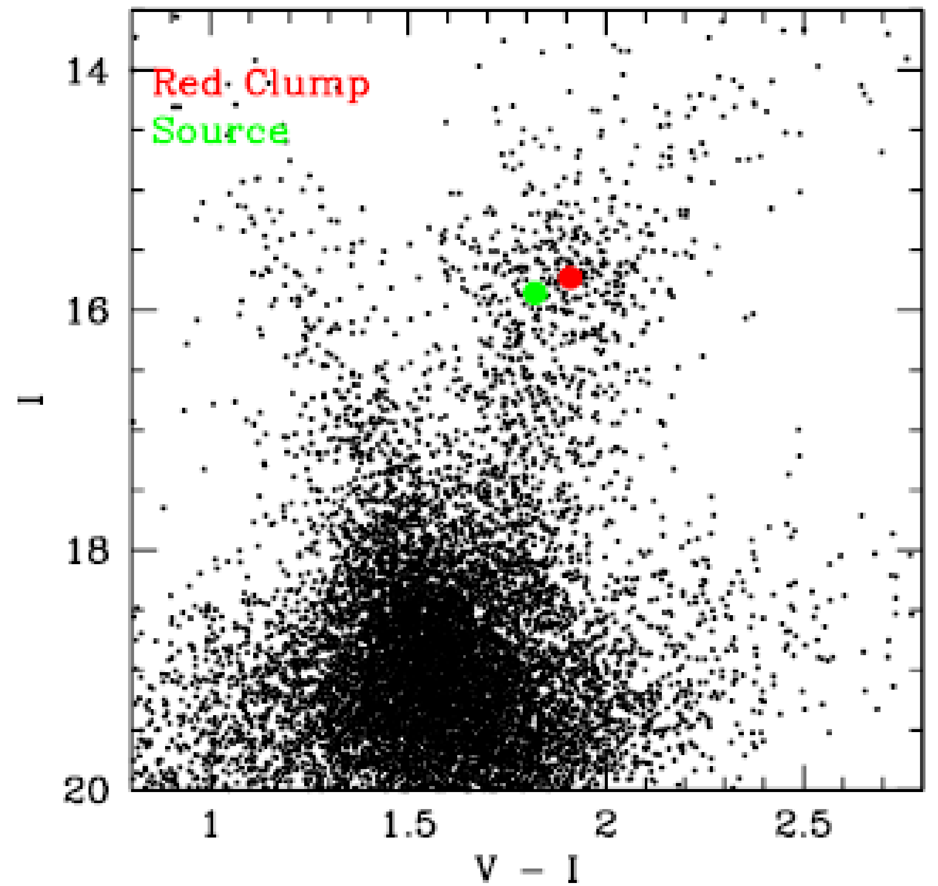




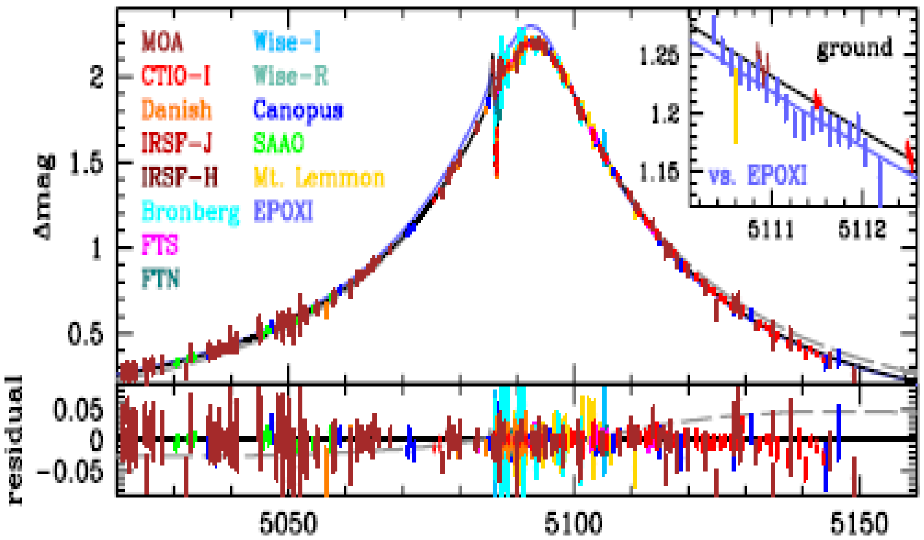
θ_* Well-measured from lightcurve

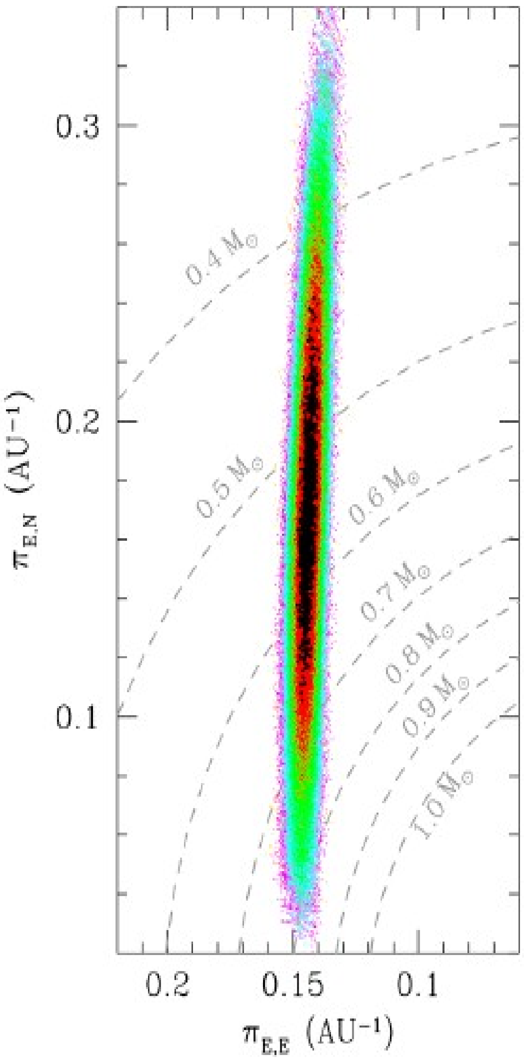
==> $\theta_E = \theta_*/\rho$





π_E semi-measured from lightcurve





Planet Lenses: + Projected Motion

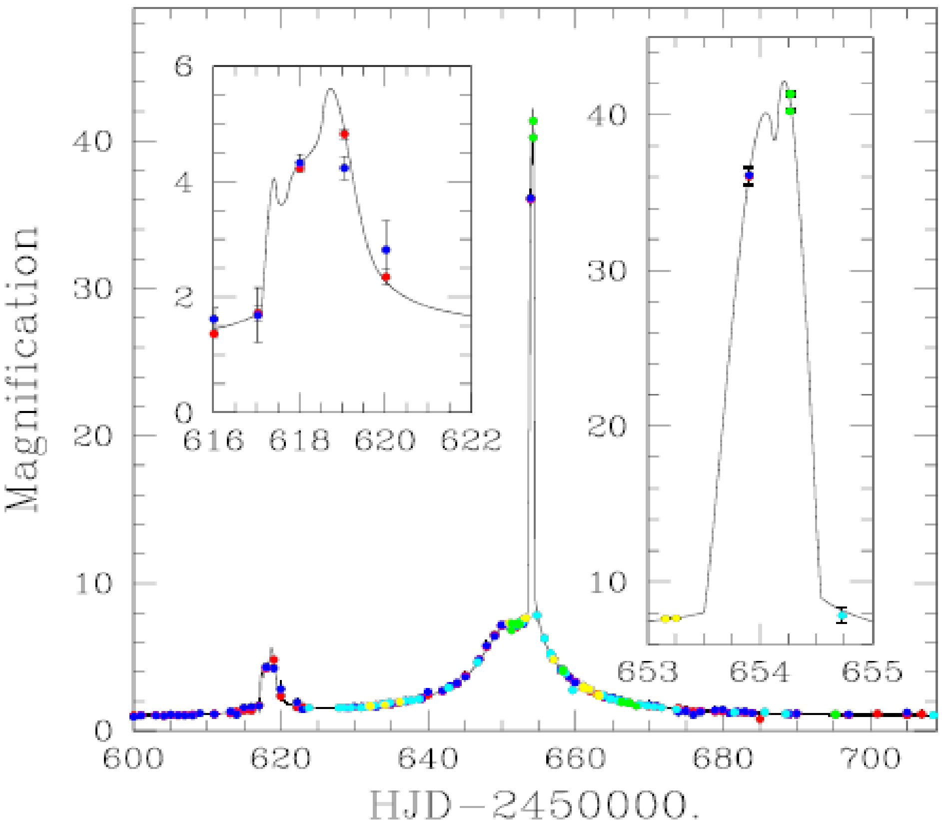
11 Features & 11 Parameters

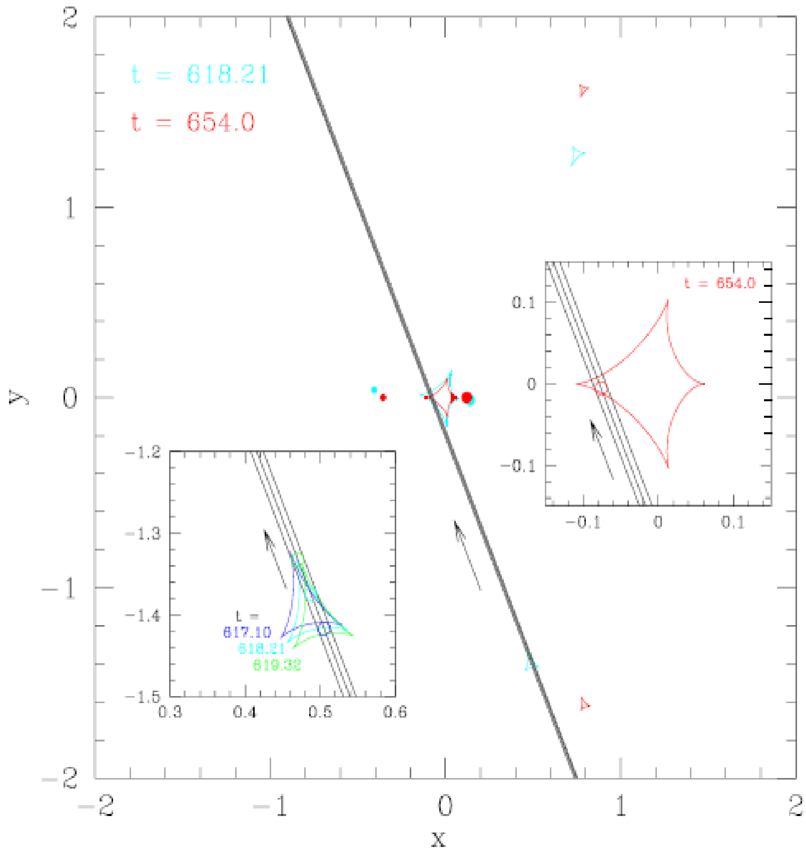
- 3 Point-Lens
- 3 Binary-Lens
- Width of Caustic Cr.
- Symmetric Distortion
- Anti-symmetric Dist.
- Rotational Motion
- Radial Motion

- t_0, u_0, t_E
- α_0, s_0, q
- $t_* = \rho * t_E$
- $\pi_{E,perp}$
- $\pi_{E,parallel}$
- $\gamma_{perp} = d\alpha/dt$
- $\gamma_{parallel} = (ds/dt)/s_0$

Macho 97-41: Obvious Orbital Motion

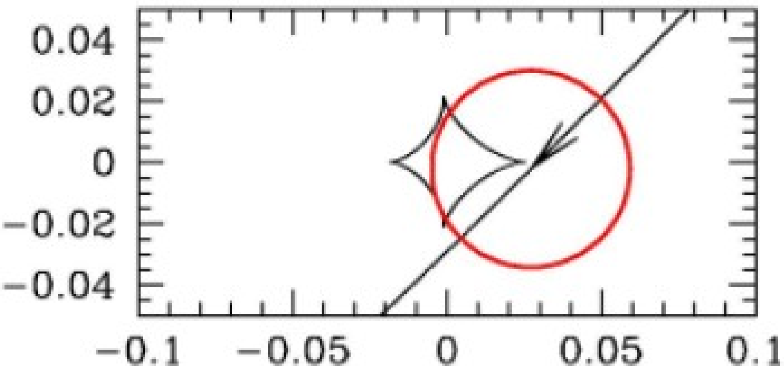
(But No Parallax)

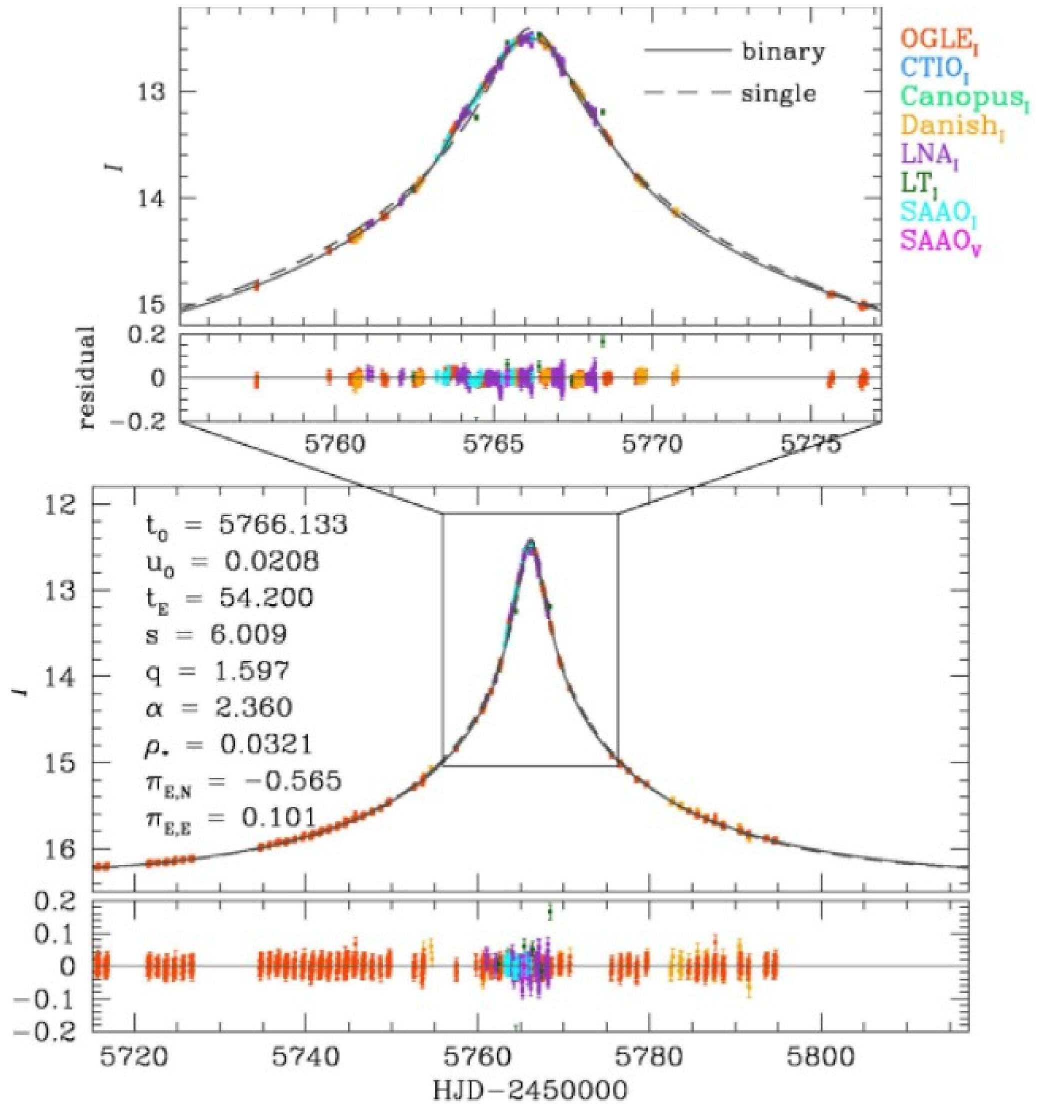




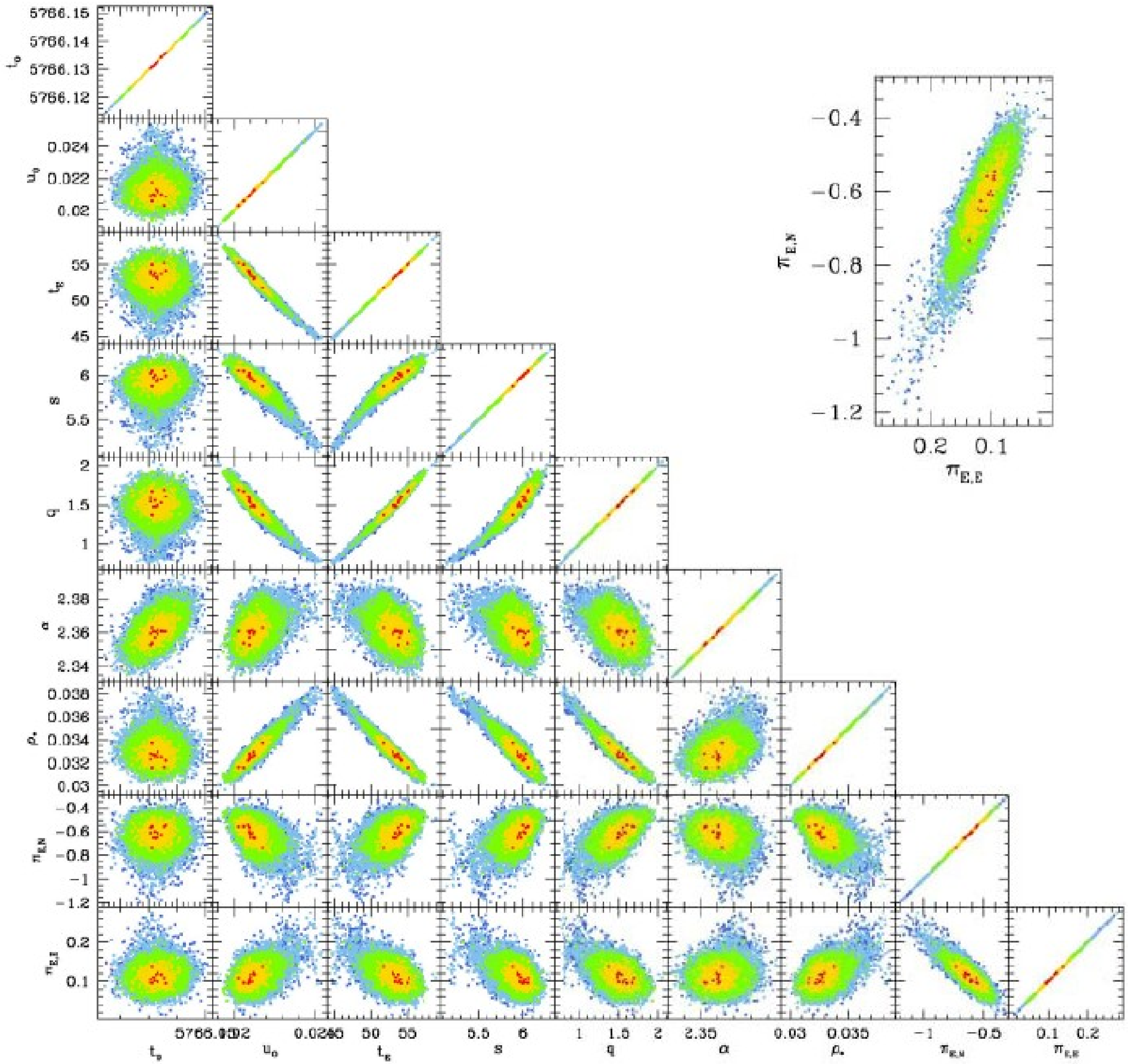
OGLE-2011-BLG-0420

Parallax + Orbital Motion





OGLE-2011-BLG-0420



OGLE-2011-BLG-0420

parameter	close	
	$u_0 > 0$	$u_0 < 0$
χ^2/dof	5427.4	5410.8
t_0 (HJD')	5766.110	5766.109
u_0	0.031	-0.030
t_E (days)	34.89	35.27
s	0.287	0.290
q	0.388	0.368
α	2.387	-2.383
$\rho_\star$	0.049	0.049
$\pi_{E,N}$	-1.03	-1.15
$\pi_{E,E}$	0.23	0.19
ds/dt (yr^{-1})	-2.44	-2.48
$d\alpha/dt$ (yr^{-1})	-8.09	7.08
KE/PE	0.36	0.32

quantity	close ($u_0 < 0$)
M_1	0.024± 0.001 $M_\odot$
M_2	0.0088 ±0.0005 $M_\odot$ (9.3 ±0.5 M_J)
D_L (kpc)	2.1 ±0.1
projected separation (AU)	0.19 ±0.01

(KE/PE)_perp: Ratio of Transverse Kinetic to Potential Energy

$$\mathrm{KE} = \frac{M_1 M_2}{M_1 + M_2}\frac{v_{\rm rel}^2}{2}; \quad \mathrm{PE} = \frac{G M_1 M_2}{r}$$

$$(\mathrm{KE})_\perp \equiv \frac{M_1 M_2}{M_1 + M_2}\frac{v_\perp^2}{2}; \quad (\mathrm{PE})_\perp \equiv \frac{G M_1 M_2}{r_\perp}$$

$$\left(\frac{\mathrm{KE}}{\mathrm{PE}}\right)_\perp = \left(\frac{\mathrm{KE}}{\mathrm{PE}}\right)\left(\frac{v_{\rm rel}}{v_\perp}\right)^2 \frac{r_\perp}{r} \le \left(\frac{\mathrm{KE}}{\mathrm{PE}}\right)$$

$$\left(\frac{\mathrm{KE}}{\mathrm{PE}}\right)_\perp = \frac{r_\perp v_{\rm rel}^2}{2GM} = \frac{r_\perp^3 \gamma^2}{2GM}$$

$$r_\perp = D_{\rm L}\theta_{\rm E} s = \frac{\mathrm{AU}\,\theta_{\rm E} s}{\pi_{\rm E}\theta_{\rm E} + \pi_s} = \frac{\mathrm{AU}\, s}{\pi_{\rm E} + \pi_s/\theta_{\rm E}}$$

$$\frac{\mathrm{AU}^3}{G M_\odot} = \left(\frac{\mathrm{yr}}{2\pi}\right)^2; \quad \frac{M}{M_\odot} = \frac{\theta_{\rm E}}{\kappa M_\odot \pi_{\rm E}} = \frac{\theta_{\rm E}/8.14\,\mathrm{mas}}{\pi_{\rm E}}$$

$$\left(\frac{\mathrm{KE}}{\mathrm{PE}}\right)_\perp = \frac{8.14}{8\pi^2}\frac{\pi_{\rm E} s^3 (\gamma\,\mathrm{yr})^2}{(\theta_{\rm E}/\mathrm{mas})(\pi_E + \pi_s/\theta_E)^3}$$

Complete Orbital Motion

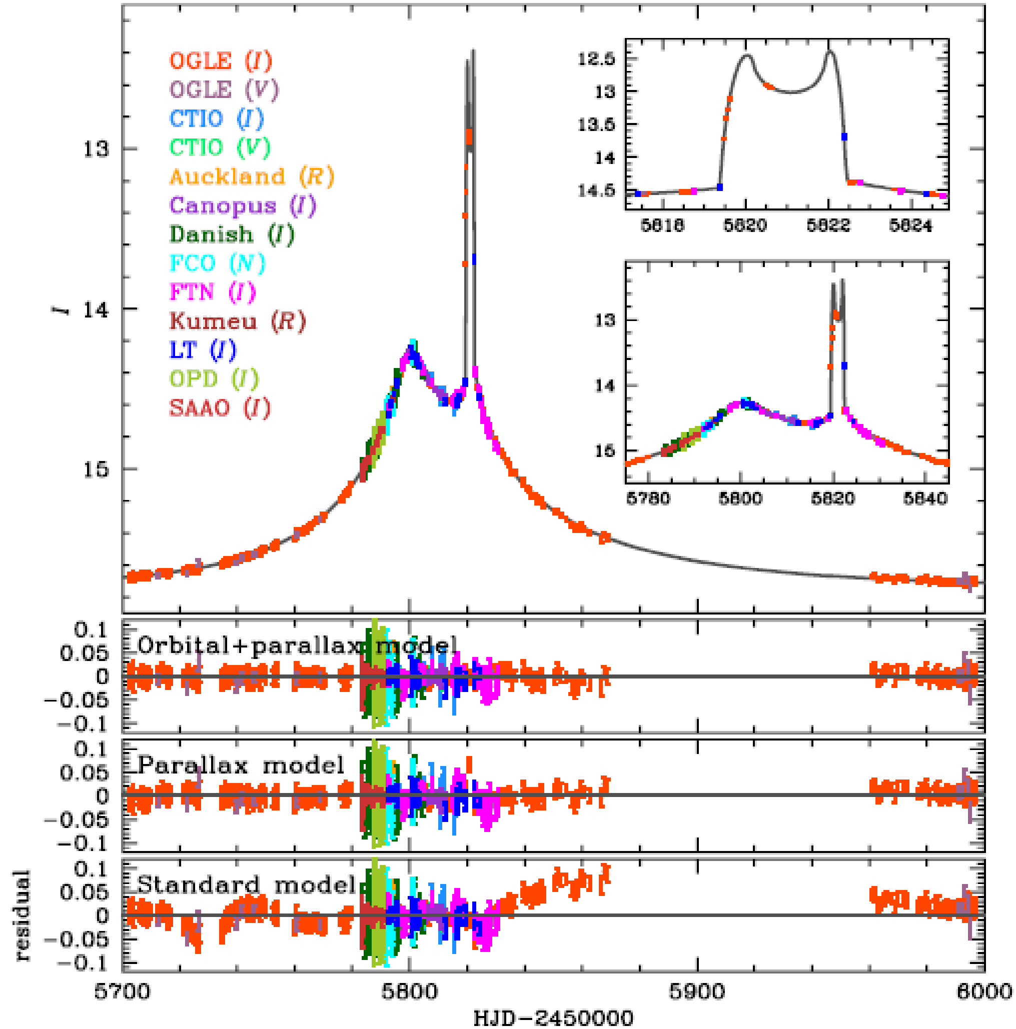
13 "Features" & 13 Parameters

- 3 Point-Lens
- 3 Binary-Lens
- Width of Caustic Cr.
- 2 Parallax
- 2 Transverse Motion
- Out-of-plane Position
- Out-of-plane Motion

- t_0, u_0, t_E
- α_0, s_0, q
- $t_* = \rho * t_E$
- $\pi_{E,perp}$, $\pi_{E,parallel}$
- γ_{perp}, $\gamma_{parallel}$
- $s_{parallel}$
- $ds_{parallel}/dt$

OGLE-2011-BLG-0417

Complete Orbital Solution



Shin et al. 2012, ApJ, 755, 91

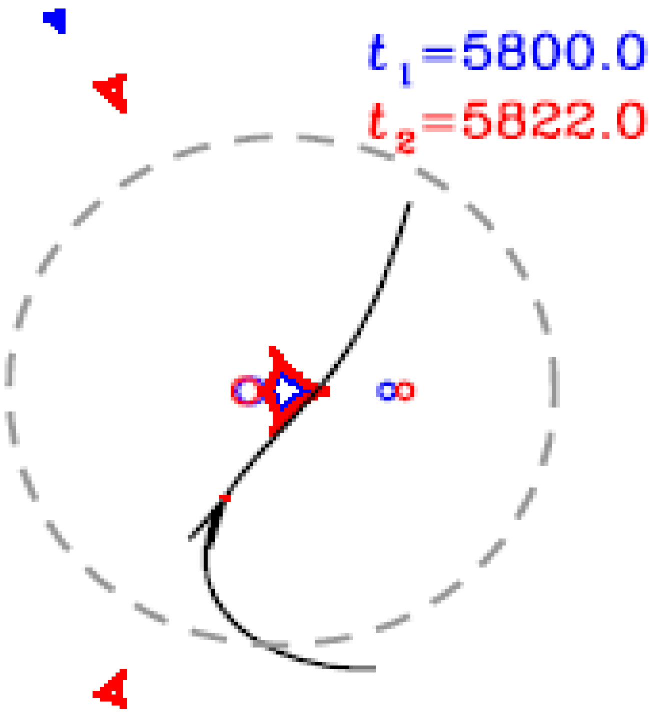
OGLE-2011-BLG-0417

Complete Orbital Solution

Parameters	Standard	Model Parallax	Orbital+Parallax
χ^2/dof	4415/2627	2391/2625	1735/2621
t_0 (HJD′)	5817.302 ± 0.018	5815.867 ± 0.030	5813.306 ± 0.059
u_0	0.1125 ± 0.0001	-0.0971 ± 0.0003	-0.0992 ± 0.0005
t_E (days)	60.74 ± 0.08	79.59 ± 0.36	92.26 ± 0.37
$s_\perp$	0.601 ± 0.001	0.574 ± 0.001	0.577 ± 0.001
q	0.402 ± 0.002	0.287 ± 0.002	0.292 ± 0.002
α (rad)	1.030 ± 0.002	-0.951 ± 0.002	-0.850 ± 0.004
ρ_* (10^{-3})	3.17 ± 0.01	2.38 ± 0.02	2.29 ± 0.02
$\pi_{E,N}$	…	0.125 ± 0.004	0.375 ± 0.015
$\pi_{E,E}$	…	-0.111 ± 0.005	-0.133 ± 0.003
$ds_\perp/dt$ (yr^{-1})	…	…	1.314 ± 0.023
$d\alpha/dt$ (yr^{-1})	…	…	1.168 ± 0.076
$s_\parallel$	…	…	0.467 ± 0.020
$ds_\parallel/dt$ (yr^{-1})	…	…	-0.192 ± 0.036

OGLE-2011-BLG-0417

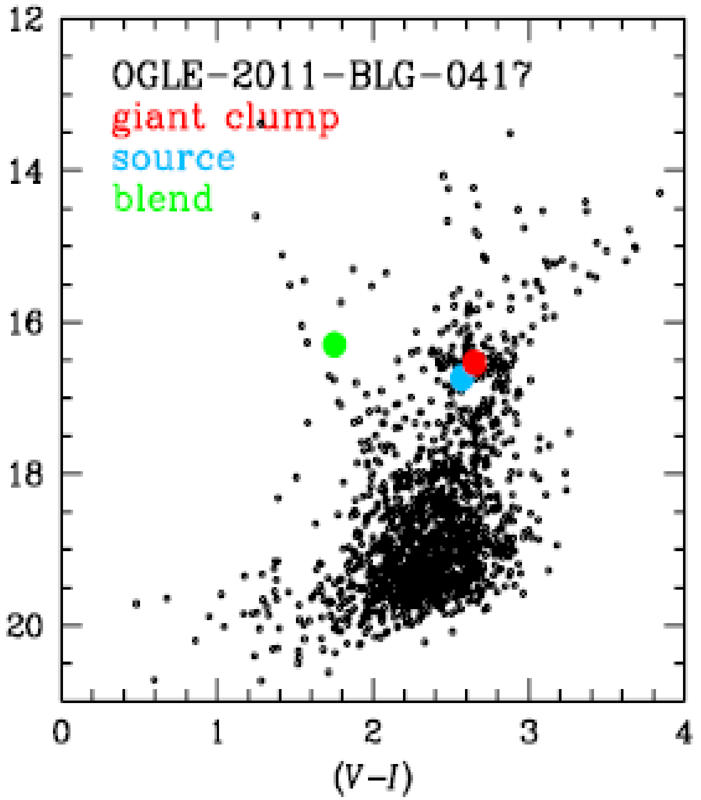
Complete Orbital Solution



Parameter	OGLE-2011-BLG-0417
$M_{\rm tot}$ ($M_\odot$)	0.74 ± 0.03
M_1 ($M_\odot$)	0.57 ± 0.02
M_2 ($M_\odot$)	0.17 ± 0.01
$\theta_{\rm E}$ (mas)	2.44 ± 0.02
μ (mas yr^{-1})	9.66 ± 0.07
$D_{\rm L}$ (kpc)	0.89 ± 0.03
a (AU)	1.15 ± 0.04
P (yr)	1.44 ± 0.06
e	0.68 ± 0.02
i (deg)	116.95 ± 1.04

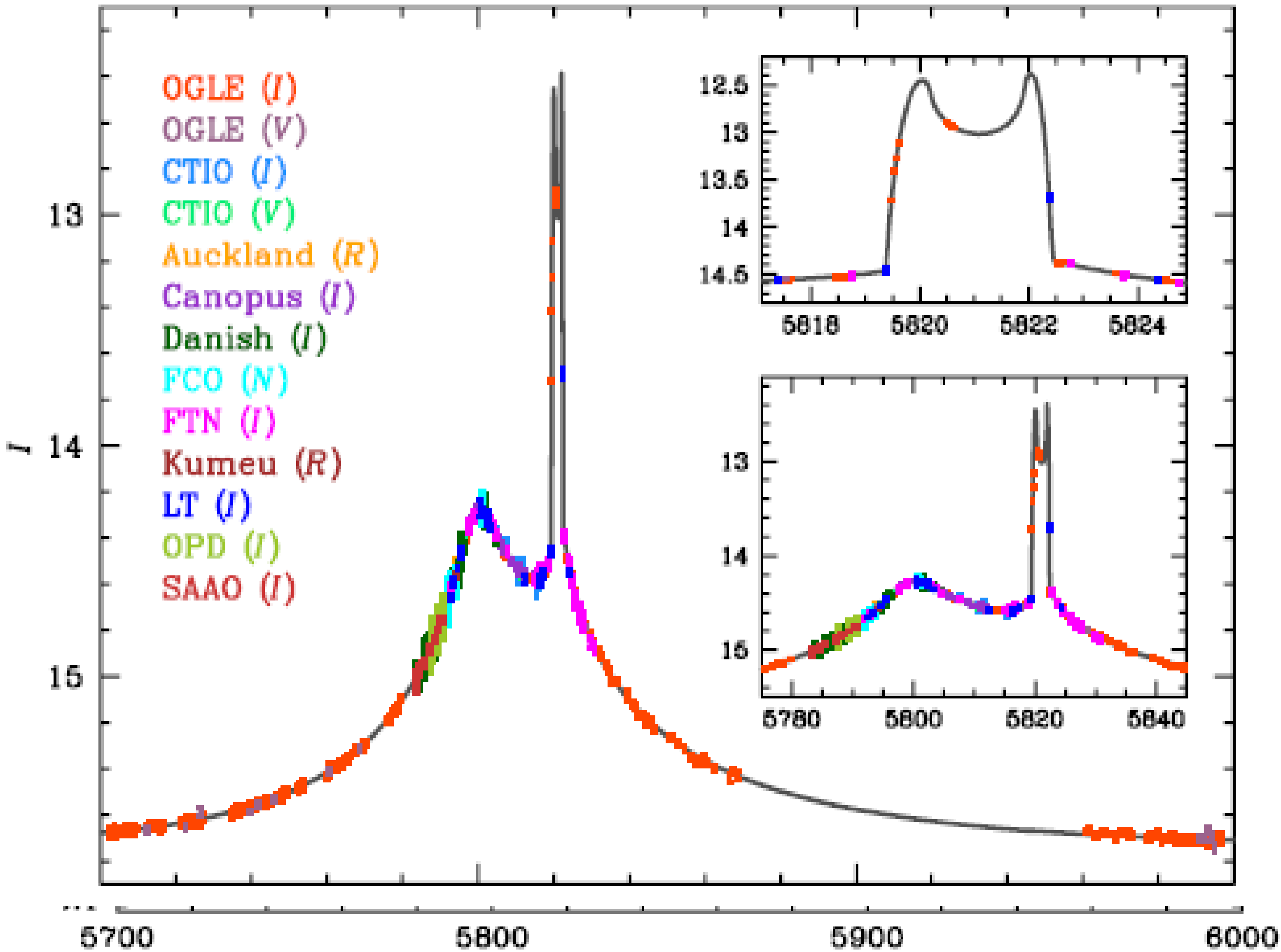
OGLE-2011-BLG-0417

Complete Orbital Solution



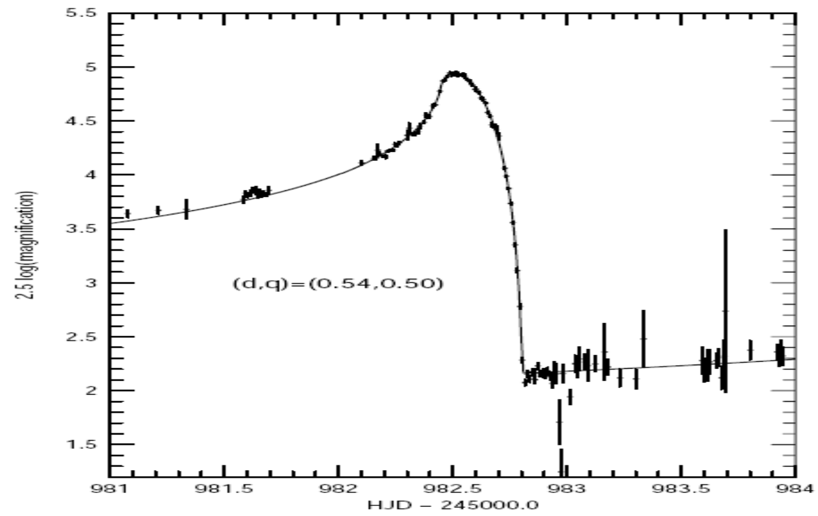
Parameter	OGLE-2011-BLG-0417
M_{tot} ($M_\odot$)	0.74 ± 0.03
M_1 ($M_\odot$)	0.57 ± 0.02
M_2 ($M_\odot$)	0.17 ± 0.01
θ_E (mas)	2.44 ± 0.02
μ (mas yr^{-1})	9.66 ± 0.07
D_L (kpc)	0.89 ± 0.03
a (AU)	1.15 ± 0.04
P (yr)	1.44 ± 0.06
e	0.68 ± 0.02
i (deg)	116.95 ± 1.04

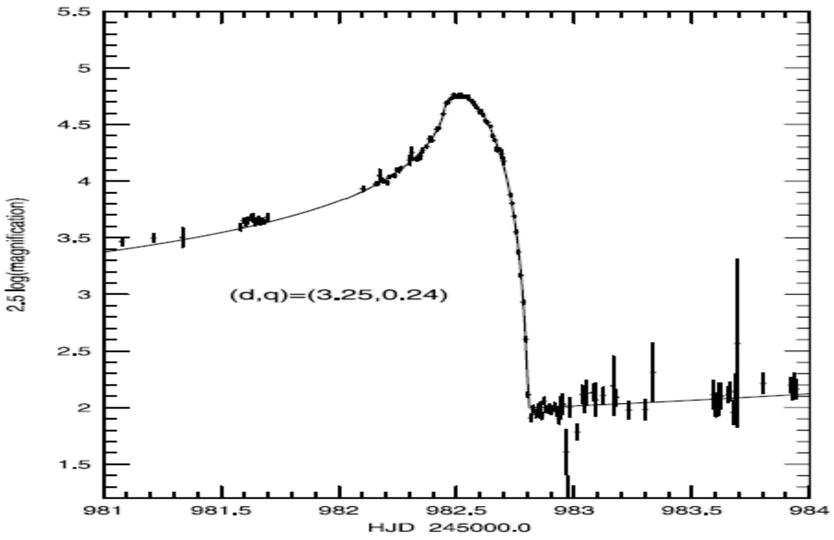
OGLE-2011-BLG-0417

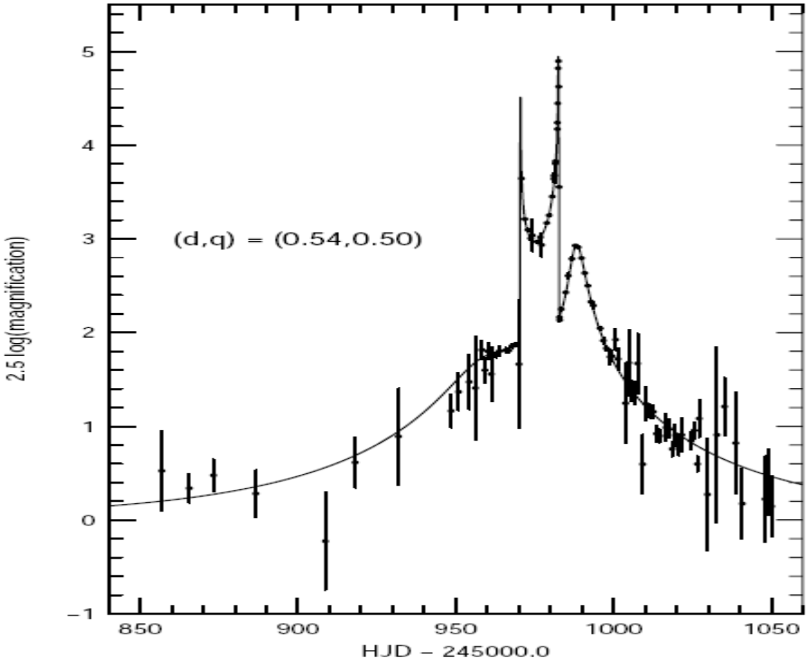


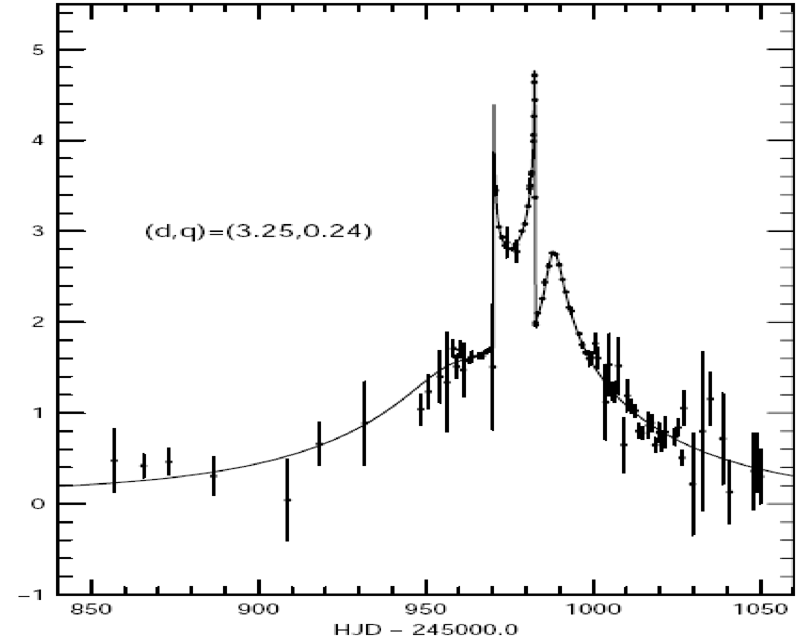
Macho-98-SMC-1

Close/Wide Binary Degeneracy

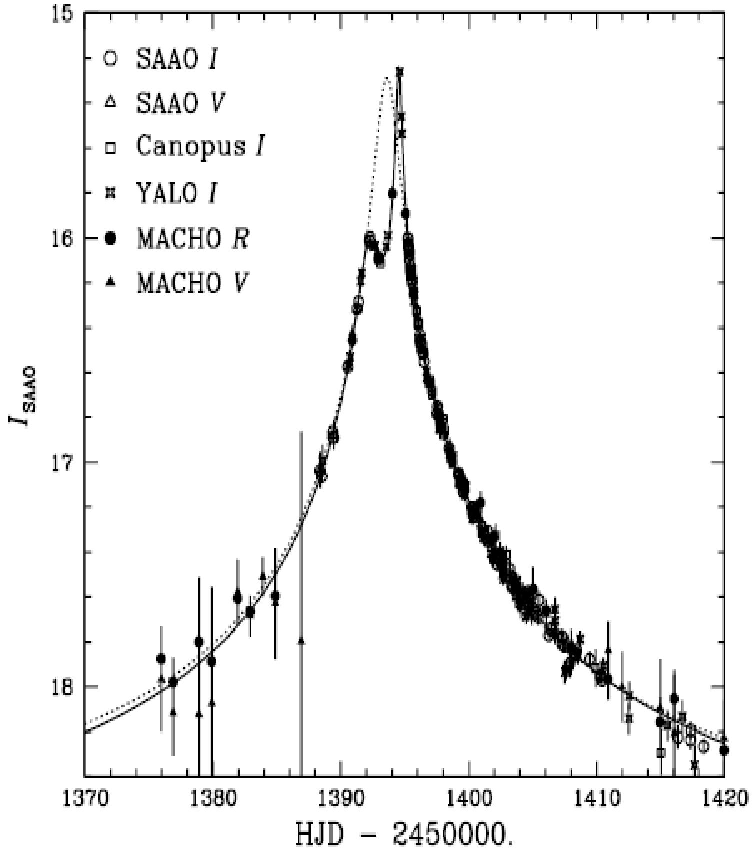


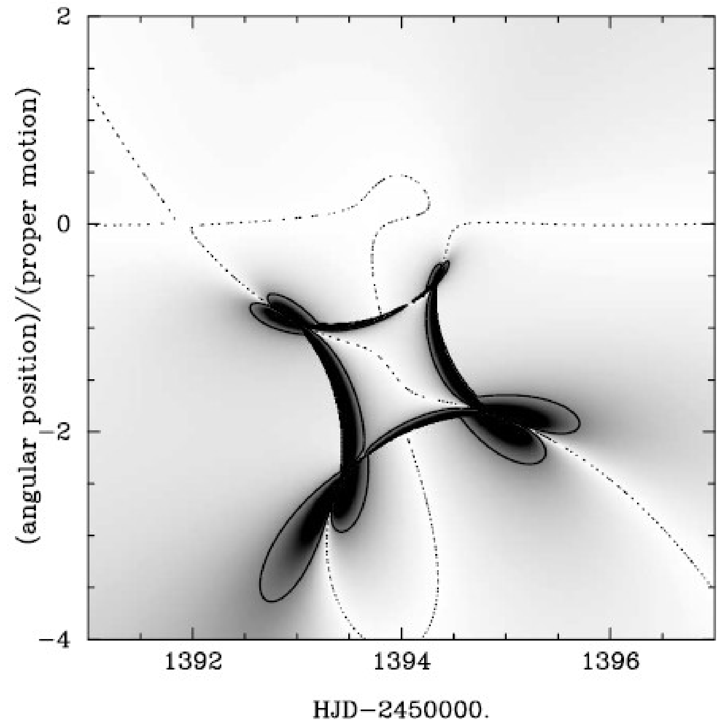


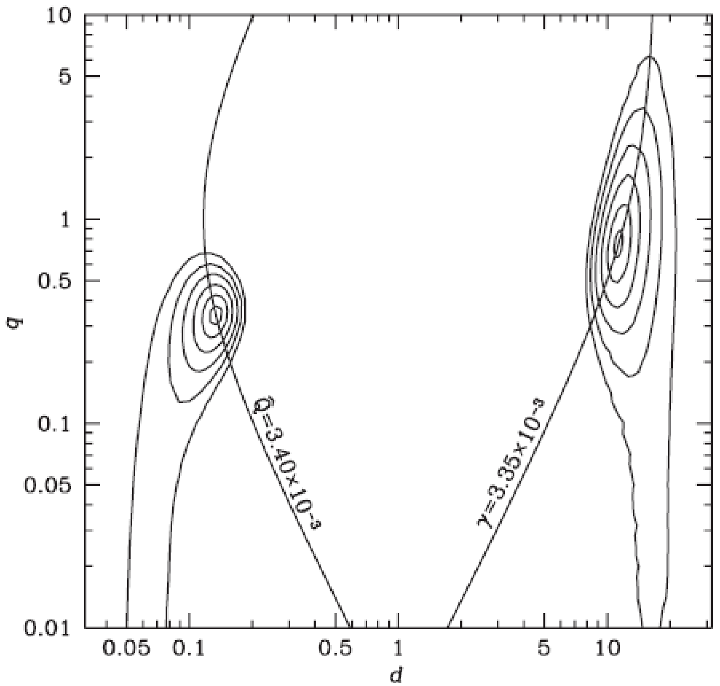




Macho-99-BLG-47







Jin An: Close/Wide Degeneracy

(At Lowest Order) [d & q]

$$\zeta = z - \frac{\epsilon_1}{\bar{z} - d_c\epsilon_2} - \frac{\epsilon_2}{\bar{z} + d_c\epsilon_1}$$
$$\approx z - \frac{1}{\bar{z}} - \frac{d_c^2\epsilon_1\epsilon_2}{\bar{z}^3} + \frac{d_c^3\epsilon_1\epsilon_2(\epsilon_1 - \epsilon_2)}{\bar{z}^4} + \cdots$$

$$\delta z_c \approx \hat{Q}\left(1 - \frac{1}{|z_0|^4}\right)^{-1}\left[\left(\frac{1}{\bar{z}_0^3} - \frac{1}{z_0^3\bar{z}_0^2}\right) + \left(\frac{1}{z_0^4\bar{z}_0^2} - \frac{1}{\bar{z}_0^4}\right)\frac{1-q_c}{1+q_c}d_c + \cdots\right]$$

$$\frac{\partial\zeta}{\partial\bar{z}} \approx \frac{1}{\bar{z}^2} + \frac{3\hat{Q}}{\bar{z}^4}\left[1 - \frac{4(1-q_c)}{3(1+q_c)}\frac{d_c}{\bar{z}} + \cdots\right]$$

$$\left|\frac{\partial\zeta}{\partial\bar{z}}\right|^2 \approx \frac{1}{|z_0|^4} + \hat{Q}\left[\frac{3|z_0|^4 - 2|z_0|^2 - 1}{|z_0|^8 - |z_0|^4}\left(\frac{1}{z_0^2} + \frac{1}{\bar{z}_0^2}\right) - \frac{4|z_0|^4 - 2|z_0|^2 - 2}{|z_0|^8 - |z_0|^4}\left(\frac{1}{z_0^3} + \frac{1}{\bar{z}_0^3}\right)\frac{1-q_c}{1+q_c}d_c + \cdots\right]$$

$$A^{-1} \approx \left|4\Delta - 2\hat{Q}\left(\frac{1}{z_0^2} + \frac{1}{\bar{z}_0^2}\right) + 3\hat{Q}\left(\frac{1}{z_0^3} + \frac{1}{\bar{z}_0^3}\right)\frac{1-q_c}{1+q_c}d_c\right|$$
$$= 4\left|(|z_0| - 1) - \hat{Q}\Re(z_0^{-2}) + \frac{3(1-q_c)}{2(1+q_c)}d_c\hat{Q}\Re(z_0^{-3})\right|$$

$$\zeta = z - \frac{1}{\bar{z}} - \frac{q_w}{\bar{z} + d_1},$$
$$\approx z - \frac{1}{\bar{z}} - \frac{q_w}{d_1} + \frac{q_w}{d_1^2}\bar{z} - \frac{q_w}{d_1^3}\bar{z}^2 + \cdots \qquad (d_w \gg |z|)$$

$$\delta z_w \approx \gamma\left(1 - \frac{1}{|z_0|^4}\right)^{-1}\left[\left(\frac{z_0}{\bar{z}_0^2} - \bar{z}_0\right) + \left(\bar{z}_0^2 - \frac{z_0^2}{\bar{z}_0^2}\right)\frac{1}{(1+q_w)^{1/2}d_w} + \cdots\right]$$

$$\frac{\partial\zeta}{\partial\bar{z}} \approx \frac{1}{\bar{z}^2} + \gamma\left[1 - \frac{2}{(1+q_w)^{1/2}}\frac{\bar{z}}{d_w} + \cdots\right]$$

$$\left|\frac{\partial\zeta}{\partial\bar{z}}\right|^2 \approx \frac{1}{|z_0|^4} + \gamma\left[\frac{|z_0|^4 + 2|z_0|^2 - 3}{|z_0|^4 - 1}\left(\frac{1}{z_0^2} + \frac{1}{\bar{z}_0^2}\right) - \frac{2|z_0|^6 + 2|z_0|^4 - 4|z_0|^2}{|z_0|^4 - 1}\left(\frac{1}{z_0^3} + \frac{1}{\bar{z}_0^3}\right)\frac{1}{(1+q_w)^{1/2}d_w} + \cdots\right]$$

$$d_c^2 d_w^2(1+q_w) = \frac{q_w}{q_c}(1+q_c)^2,$$

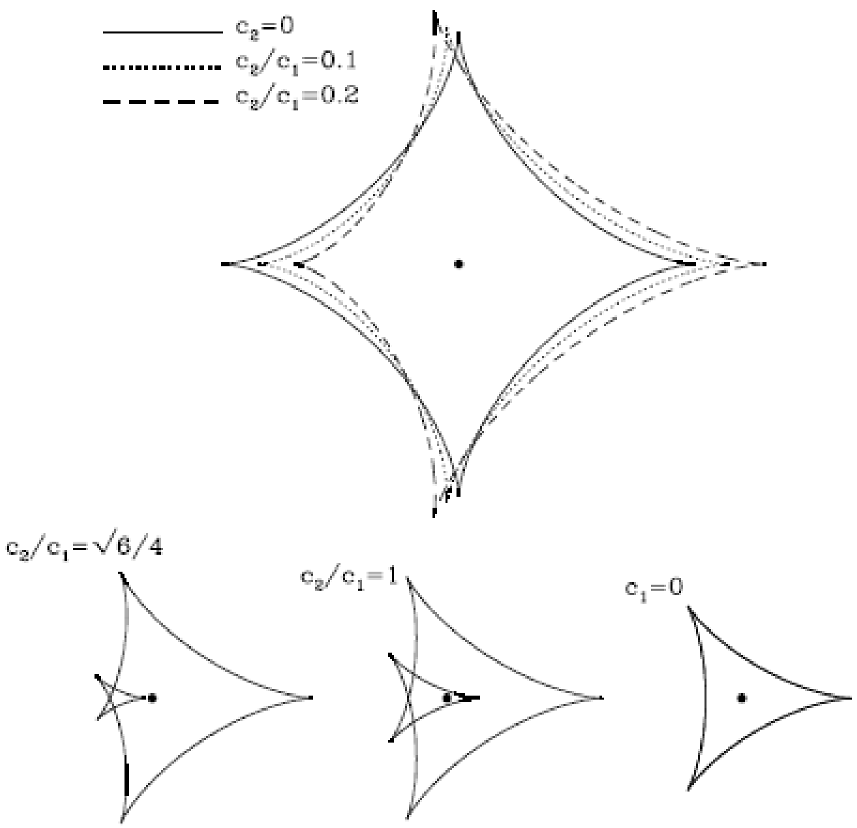
$$A^{-1} \approx \left|4\Delta - 2\gamma\left(\frac{1}{z_0^2} + \frac{1}{\bar{z}_0^2}\right) + 3\gamma\left(\frac{1}{z_0^3} + \frac{1}{\bar{z}_0^3}\right)\frac{1}{(1+q_w)^{1/2}d_w}\right|$$
$$= 4\left|(|z_0| - 1) - \gamma\Re(z_0^{-2}) + \frac{3}{2(1+q_w)^{1/2}}\frac{\gamma}{d_w}\Re(z_0^{-3})\right|$$

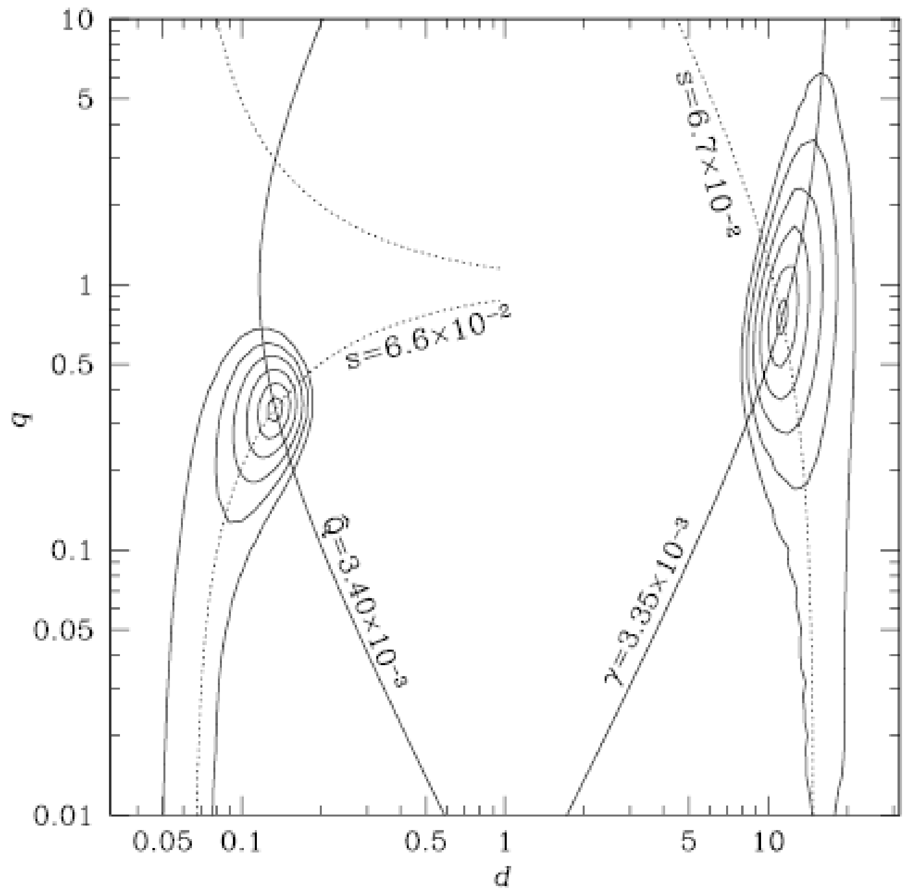
$$d_c d_w(1+q_w)^{1/2} = \frac{1+q_c}{1-q_c},$$

Jin An: Wide/Close Degeneracy

(At Second Order)

[Shape Parameter: s = c_2/c_1]



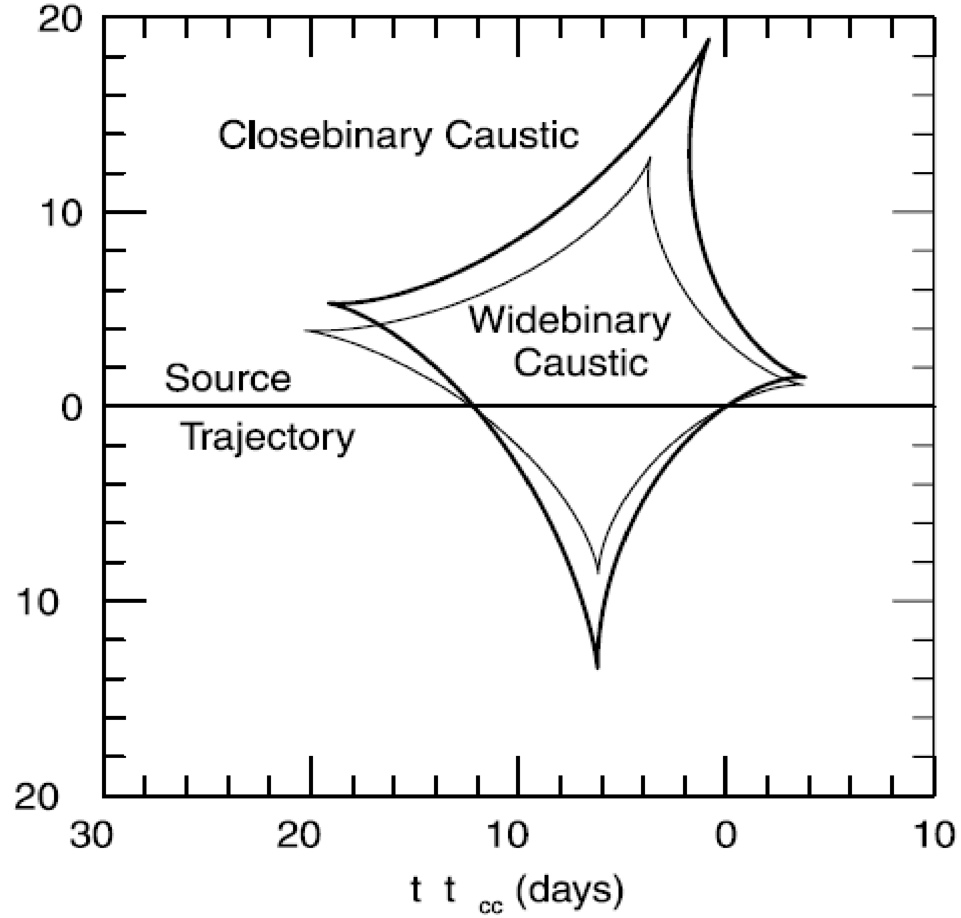


Different caustics -> Same lightcurve

An 2005, MNRAS, 356, 1409

$$q_c \to 1 - \frac{\sqrt{1+4q_w} - 1}{2q_w}$$

$$b_c \to b_w^{-1} \sqrt{\frac{1+4q_w}{1+q_w}}$$

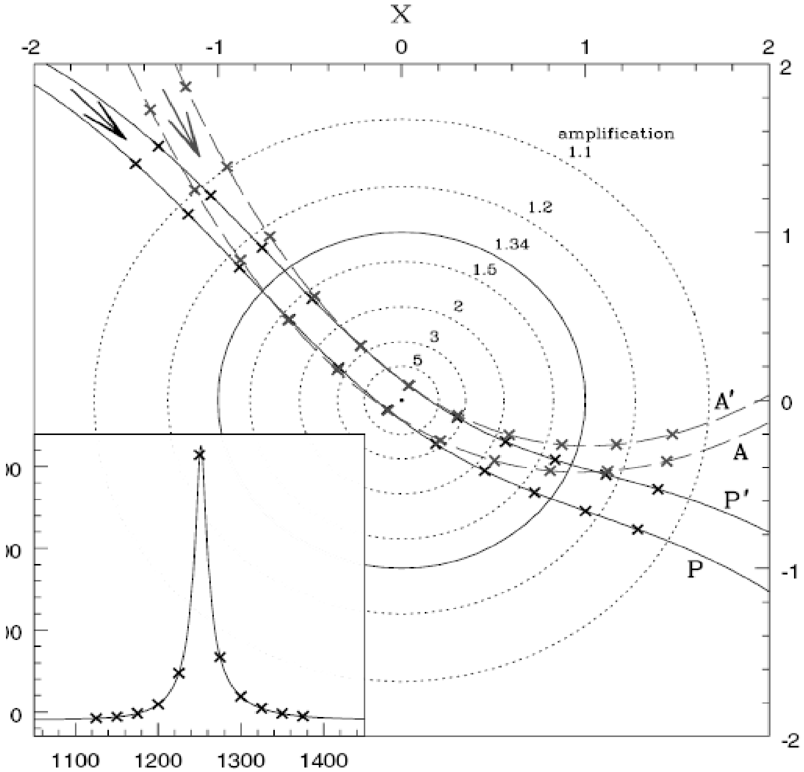


Ecliptic Degeneracy

Begins in 'constant acceleration' model

- u_0 --> -u_0

- Smith, Mao, & Paczynski (2003)



Ecliptic Degeneracy

Embedded in 'jerk parallax' formalism

- u_0 --> -u_0
- |u_0|<<1 ==> jerk-par

- SMP (2003)
- Gould (2004)

$$\pi'_{\mathrm{E},\parallel} = \pi_{\mathrm{E},\parallel}, \qquad \pi'_{\mathrm{E},\perp} = -(\pi_{\mathrm{E},\perp} + \pi_{j,\perp}),$$

$$\pi_{j,\perp} = -\frac{4}{3}\frac{\mathrm{yr}}{2\pi t_{\mathrm{E}}}\frac{\sin\beta_{\mathrm{ec}}}{\left(\cos^2\psi\sin^2\beta_{\mathrm{ec}} + \sin^2\psi\right)^{3/2}}$$

Ecliptic Degeneracy

Jiang et al.: Exact Degenercy (β_{ec}=0)

- u_0 --> -u_0
- |u_0|<<1 ==> jerk-par
- $(u_0,\pi_{E,perp})\text{->}-(u_0,\pi_{E,perp})$

- SMP (2003)
- Gould (2004)
- Jiang et al. (2004)

$$\pi_{j,\perp} = -\frac{4}{3}\frac{\mathrm{yr}}{2\pi t_{\mathrm{E}}}\frac{\sin\beta_{\mathrm{ec}}}{\left(\cos^2\psi\sin^2\beta_{\mathrm{ec}} + \sin^2\psi\right)^{3/2}}$$

Ecliptic Degeneracy

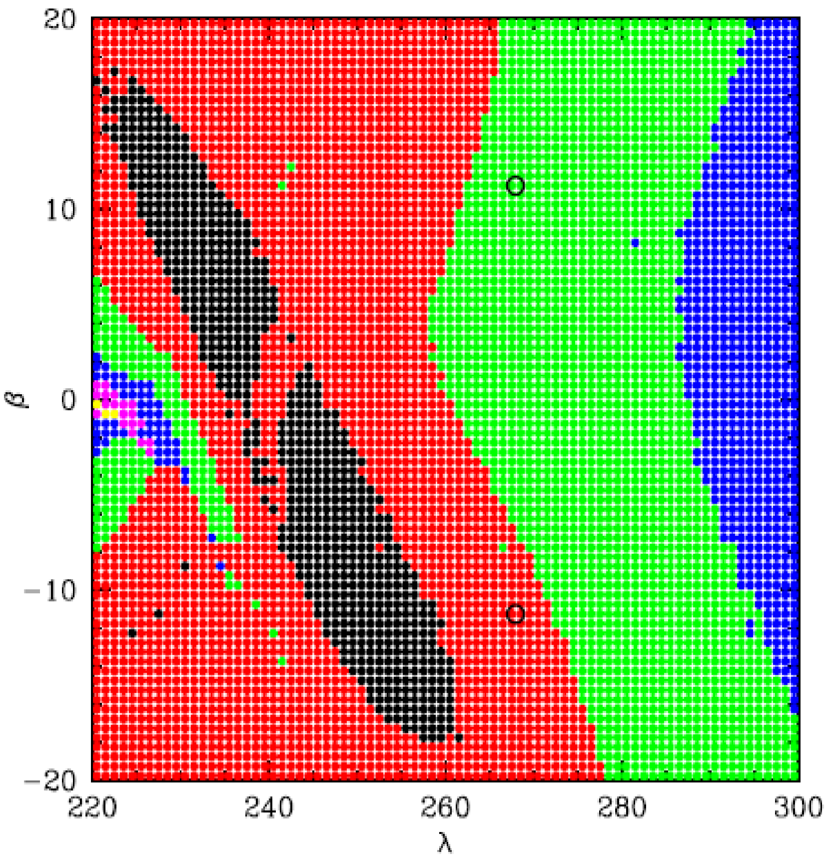
Skowron et al. 2011, ApJ, 738,87

generalize to binaries

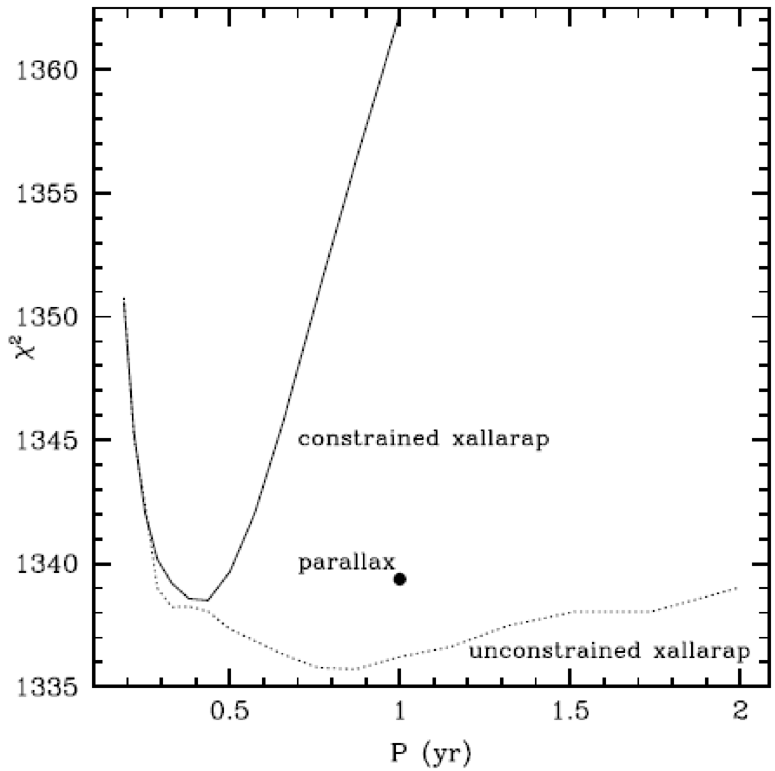
- u_0 --> -u_0
- |u_0|<<1 ==> jerk-par
- $(u_0,\pi_{E,perp}) \to -(u_0,\pi_{E,perp})$
- $(u_0,\pi_{E,perp},\alpha) \to -(u_0,\pi_{E,perp},\alpha)$
- $(u_0,\pi_{E,perp},\alpha_0,d\alpha/dt) \to -(u_0,\pi_{E,perp},\alpha_0,d\alpha/dt)$

- SMP (2003)
- Gould (2004)
- Single
- Static Binary
- Rotating Binary

Xallarap vs. Parallax



Xallarap vs. Parallax



Point-lens magnfication

Start: Binary-Lens Equation

$$\mathbf{u} - \mathbf{y} = -\frac{\mathbf{y} - \mathbf{y}_L}{|\mathbf{y} - \mathbf{y}_L|^2}$$

$$\mathbf{y}_L = 0 \rightarrow \mathbf{u} - \mathbf{y} = -\frac{\mathbf{y}}{y^2} \Longrightarrow u - y = -\frac{1}{y}$$

$$\Longrightarrow (y - u)y = 1 \Longrightarrow (\theta_I - \theta_S)\theta_I = \theta_{\mathrm{E}}^2$$

$$\mathbf{u} = \mathbf{y} - \sum_i \epsilon_i \frac{\mathbf{y} - \mathbf{y}_{m,i}}{|\mathbf{y} - \mathbf{y}_{m,i}|^2} \qquad \epsilon_i \equiv \frac{m_i}{M_{\mathrm{tot}}}$$

$$\zeta = z - \sum_i \frac{\epsilon_i}{\bar{z} - \bar{z}_{m,i}}$$

$$\zeta \equiv u_1 + i u_2 \qquad z \equiv y_1 + i y_2$$

Why is this a Fifth-Order Equation?

$$\zeta = z - \sum_i \frac{\epsilon_i}{\bar{z} - \bar{z}_{m,i}}$$

$$\zeta \equiv u_1 + iu_2; \qquad z \equiv y_1 + iy_2$$

$$z = \zeta + \frac{\epsilon_1}{\bar{z} - \bar{z}_1} + \frac{\epsilon_2}{\bar{z} - \bar{z}_2}$$

$$\bar{z} = \bar{\zeta} + \frac{\epsilon_1}{z - z_1} + \frac{\epsilon_2}{z - z_2}$$

$$(z - \zeta)(\bar{z} - \bar{z}_1)(\bar{z} - \bar{z}_2) = \epsilon_1(\bar{z} - \bar{z}_2) + \epsilon_2(\bar{z} - \bar{z}_1)$$

$$(z - \zeta)\left(\bar{\zeta} + \frac{\epsilon_1}{z - z_1} + \frac{\epsilon_2}{z - z_2} - \bar{z}_1\right)\left(\bar{\zeta} + \frac{\epsilon_1}{z - z_1} + \frac{\epsilon_2}{z - z_2} - \bar{z}_2\right)$$

$$= \left(\bar{\zeta} + \frac{\epsilon_1}{z - z_1} + \frac{\epsilon_2}{z - z_2} - \bar{z}_2\right)\epsilon_1 + \left(\bar{\zeta} + \frac{\epsilon_1}{z - z_1} + \frac{\epsilon_2}{z - z_2} - \bar{z}_1\right)\epsilon_2$$

Magnification (A): For each image, i

1) Check that it solves lens equation

2) Calculate A_i from determinant

$$\partial\zeta_i = \sum_k \frac{\epsilon_k}{(\bar{z} - \bar{z}_k)^2}$$

$$A_i = \frac{1}{1 - |\partial\zeta_i|^2}$$

$$A = \sum_i |A_i|$$

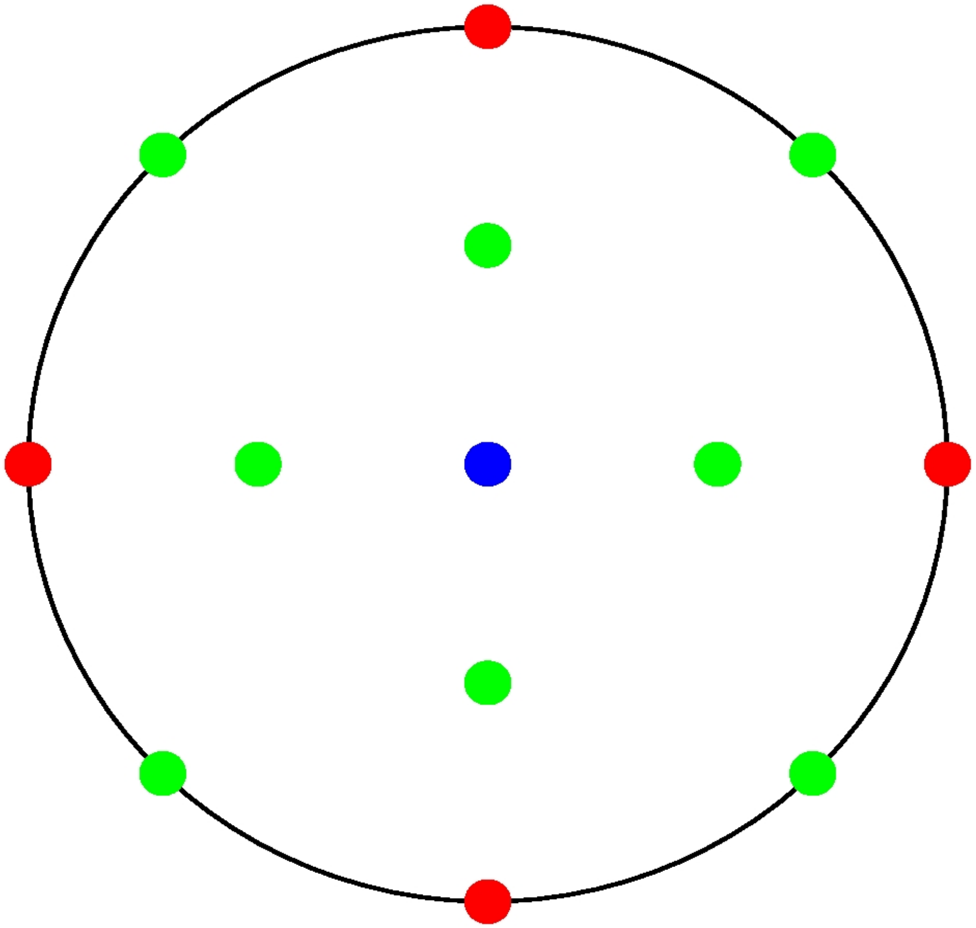
Quadrupole/Hexadecapole

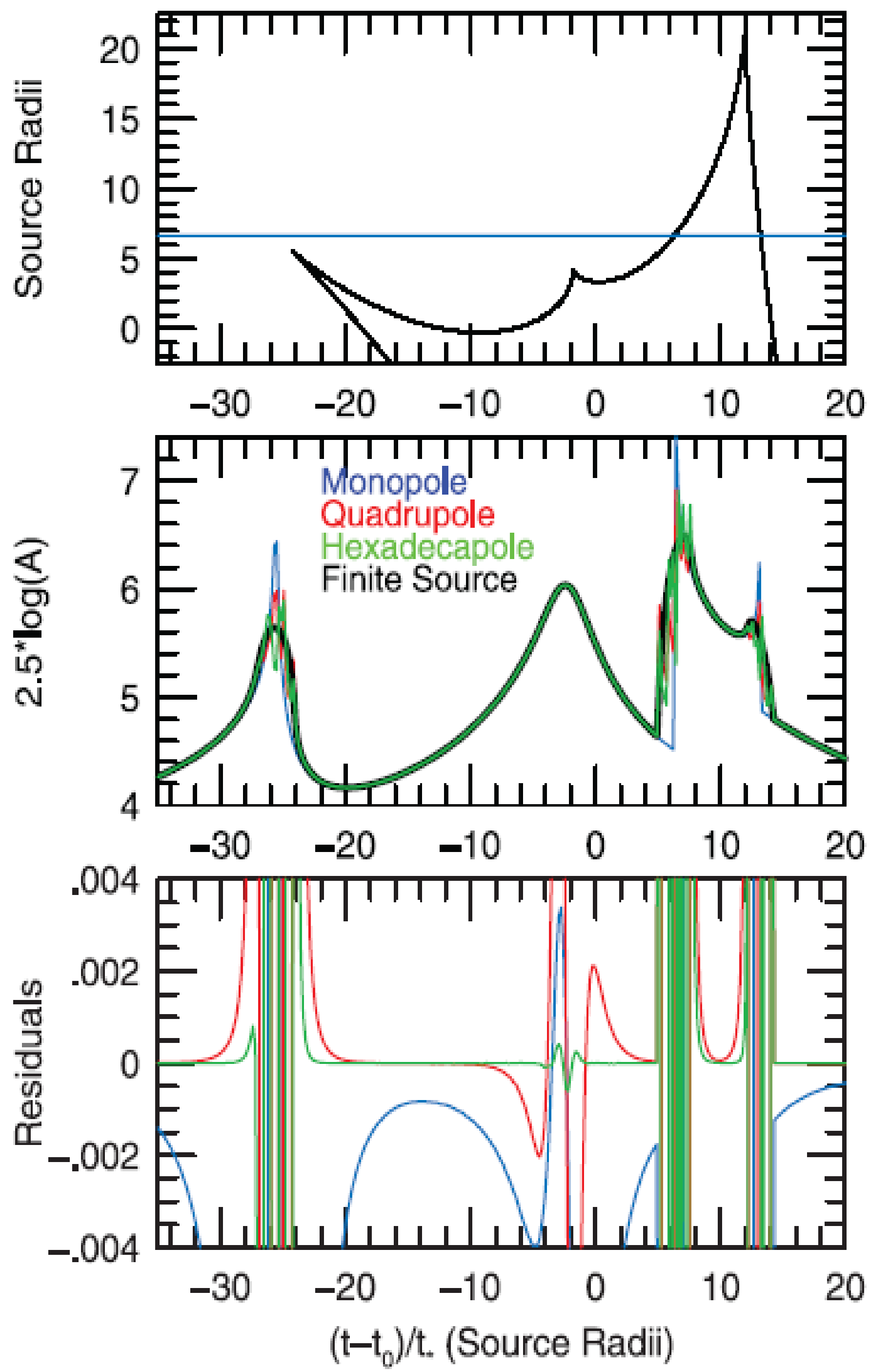
Pejcha & Heyrovsky (2009)

Gould (2008)

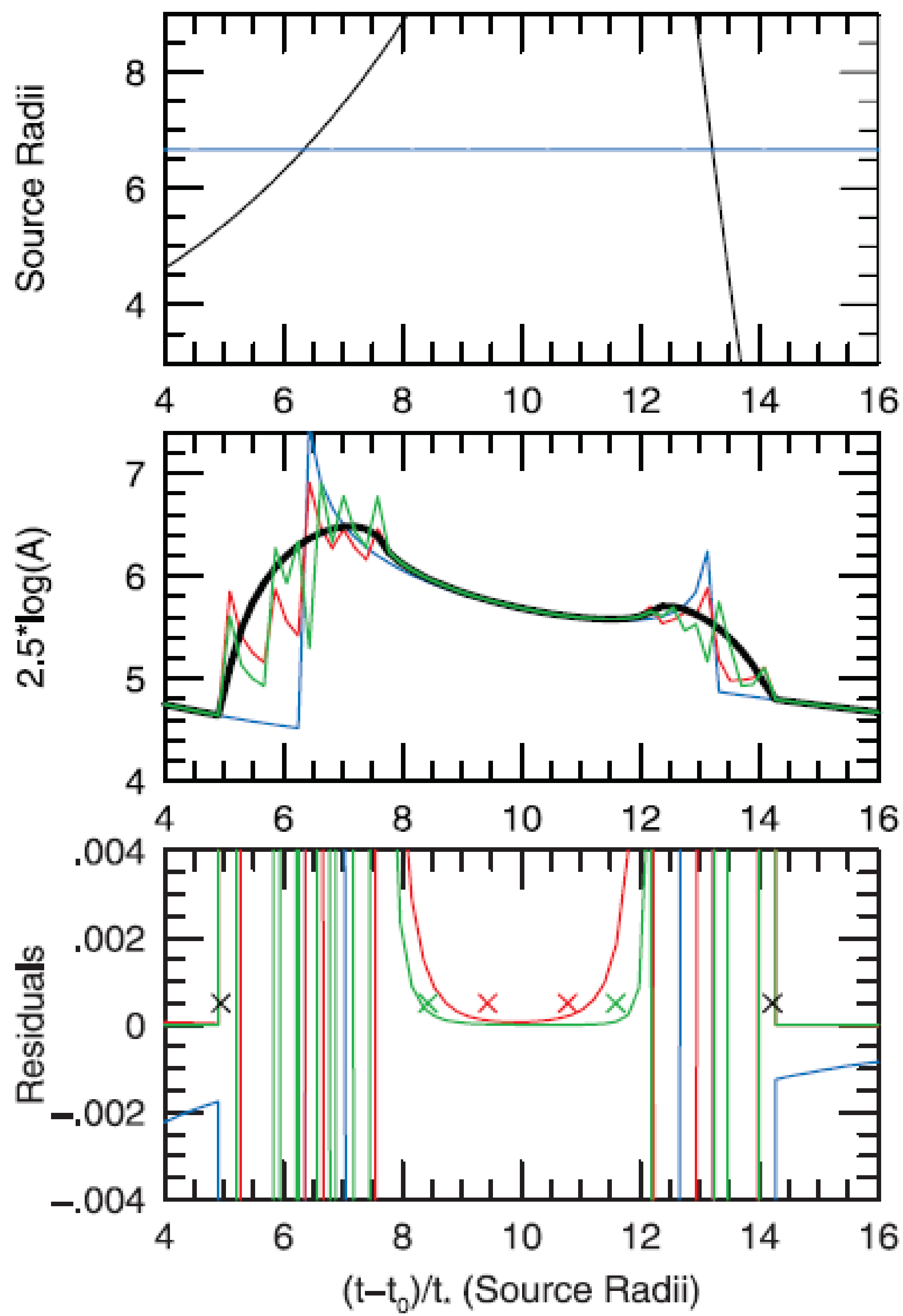
- Pure gradient: Monopole (pt lens)
- Mild curvature: Quadrupole
- Stronger curvature: Hexadecapole
- Extreme curvature: New Method

Monopole/Quadrupole/Hexadecapole

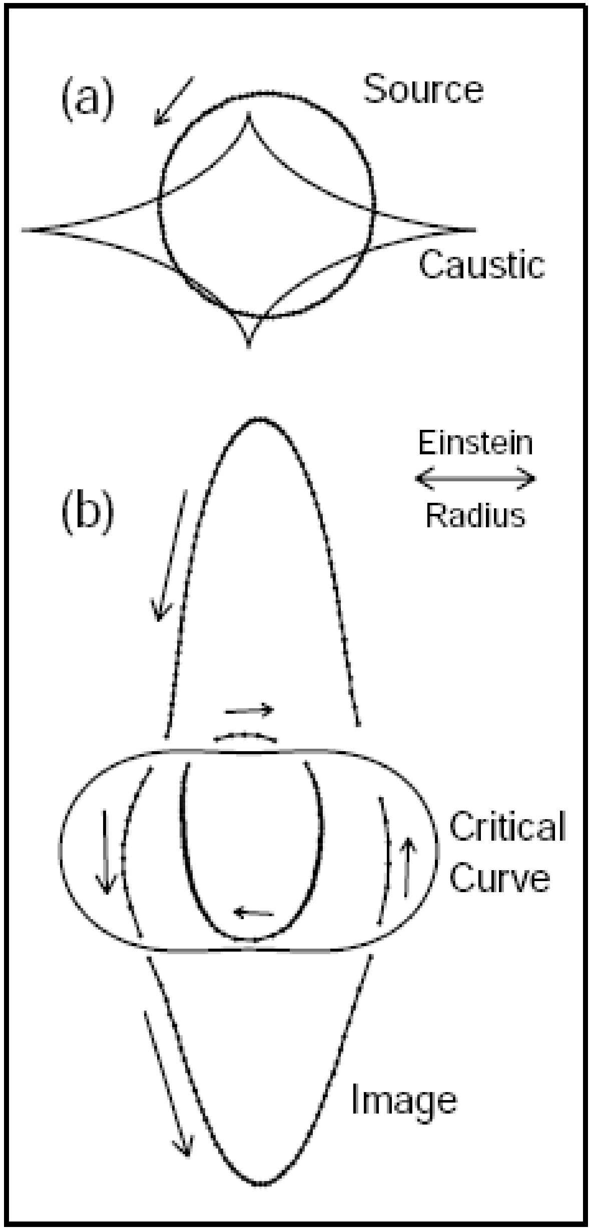




Source Radii
2.5*log(A)
Monopole
Quadrupole
Hexadecapole
Finite Source
Residuals
(t–t0)/t. (Source Radii)


Source Radii
2.5*log(A)
Residuals
(t−t0)/t. (Source Radii)

Contour Integration: Gould & Gaucherel (1997)



$$A = \sum_{i=1}^{n} \sum_{j'} p_j (u_{i-1,j} \times u_{i,j}) \Big/ \sum_{i=1}^{n} s_{i-1} \times s_i \, ,$$

Contour Integration

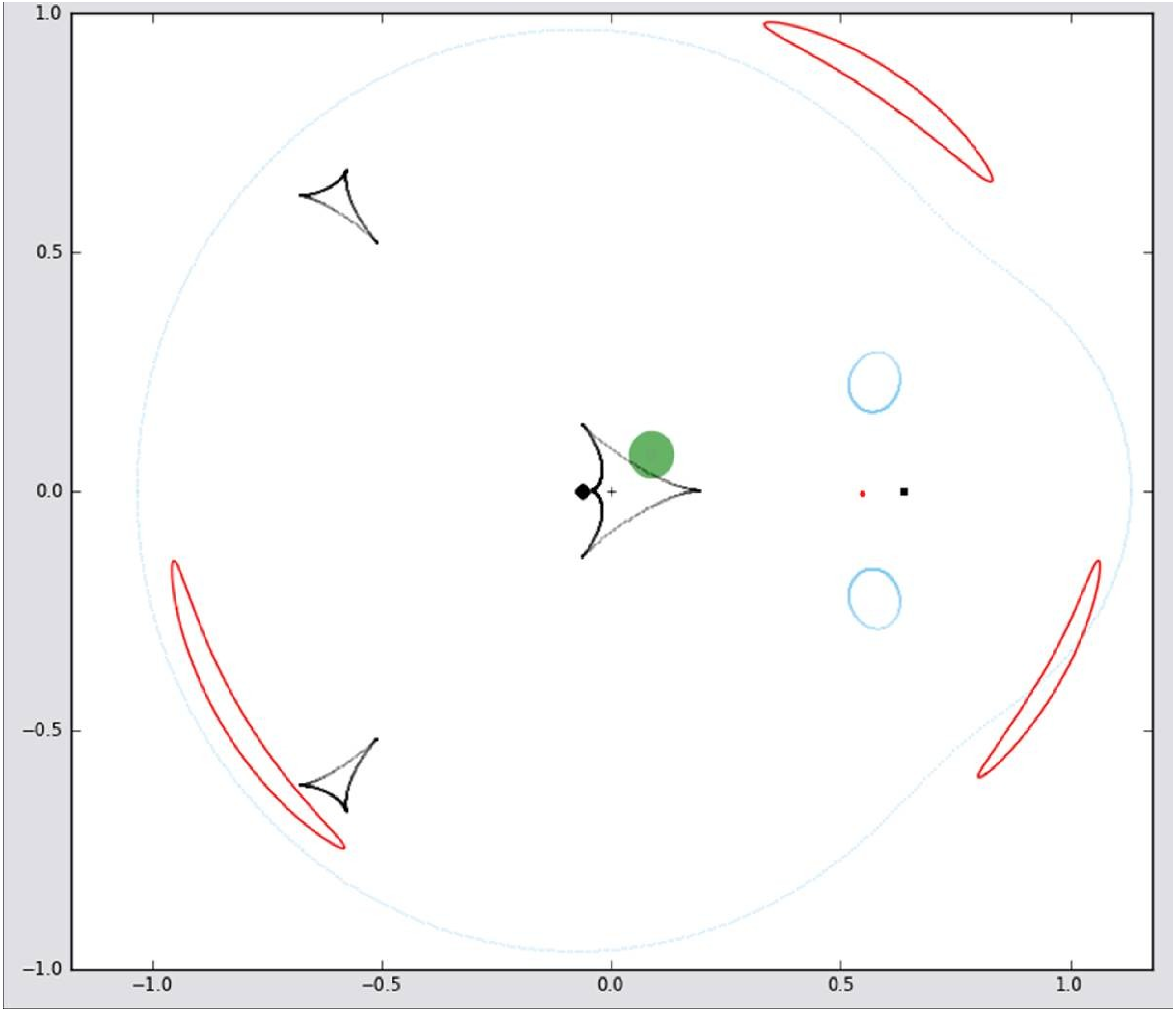
- Fastest INDIVIDUAL FS calculation
- Disadvantage 1: Limb Darkening -> many cont.
- Disadvantage 2: Cusp lens-solver hang-ups
- Neither fatal (see below)
- Major improvements from Bozza (2010)
  - MNRAS 408 2188 (code not public)

Inverse Ray Shooting (General)

$$u = y - \sum_i \epsilon_i \frac{y - y_{m,i}}{|y - y_{m,i}|^2} \qquad \epsilon_i \equiv \frac{m_i}{M_{tot}}$$

- do i=1,n
- pick: y_i
  - image plane point
- calculate: u_i
  - source plane point
- store (u_i <--> y_i)
- enddo

- Pick source boundary
- Examine each pt u_i
  - in: weight by LD
  - out: discard
- Sum up weights

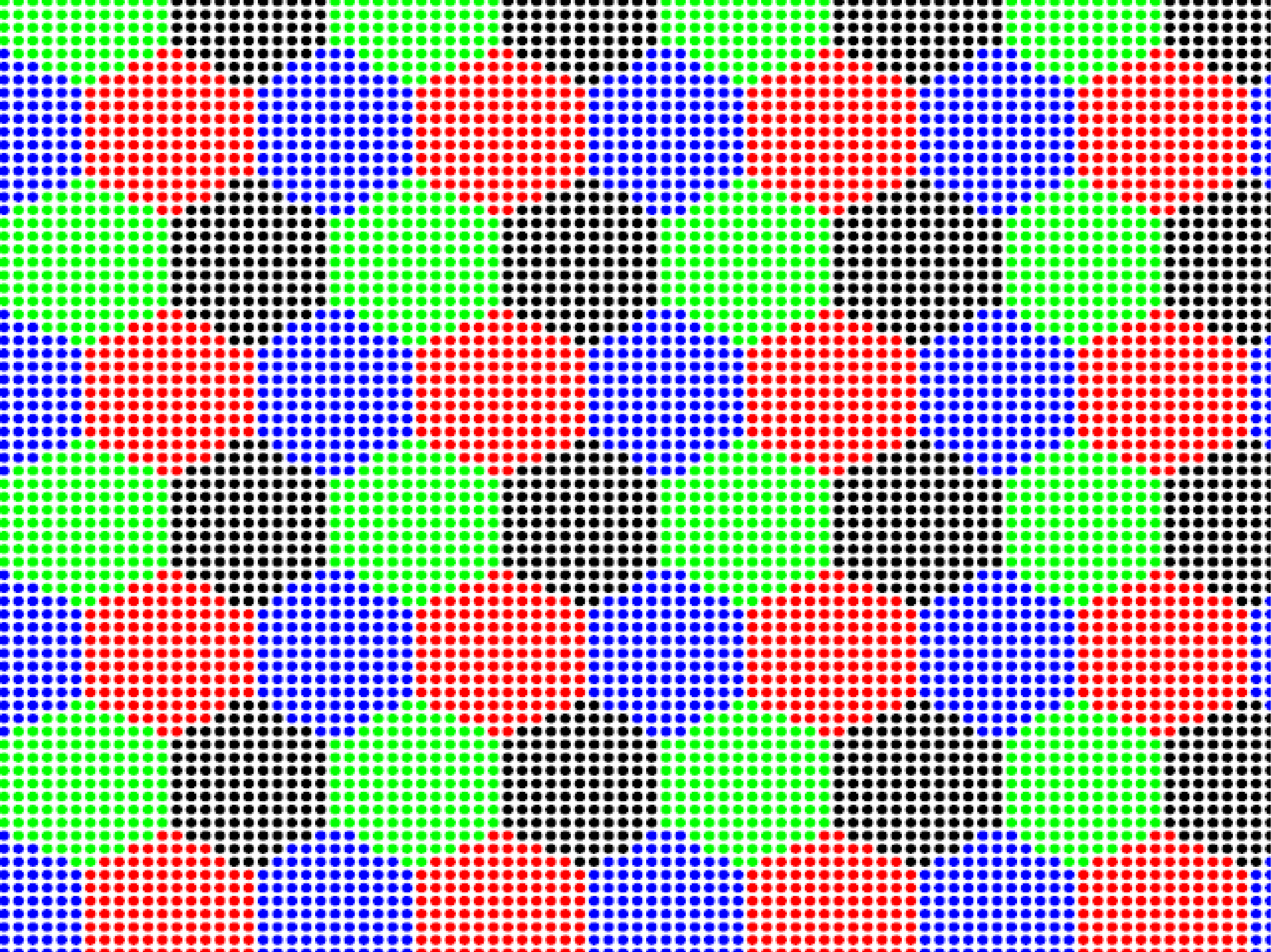

1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0

Inverse Ray Shooting

- Advantages
  - Automatically includes LD
  - Always works
- Disadvantage
  - Very expensive to shoot lens plane

IRS 1: Map-Making

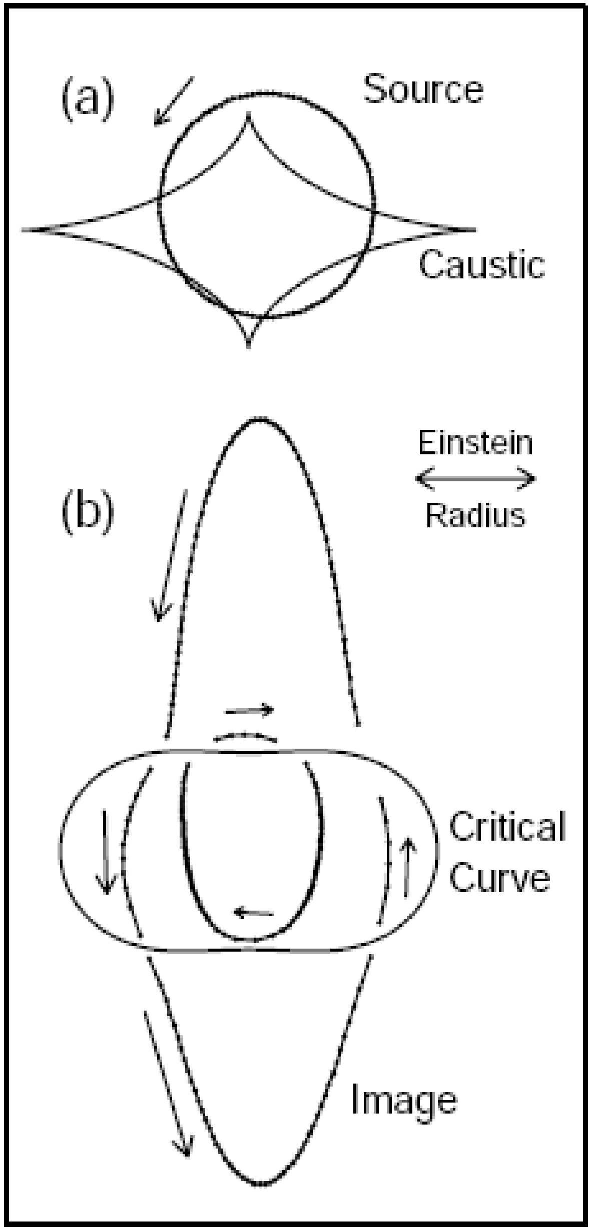
- Dong et al. 2006 ApJ 642 842
- Shoot entire annulus relevant to event
- Store rays & hex-tiles on source plane
- Use hex-tiles for interior, rays for edge
- All (s,q) light curves use ONE map
- Disadvantage: requires fixed “s”



IRS 2A: Loop Linking

- Dong et al. 2006 ApJ 642 842
- Make contour (as in Gould+Gaucherel)
  - Except slightly bigger
- Shoot rays within images (IRS general)
- Advantage: avoids contour problems
- Disadvantage: costs more than contour

Contour Integration: Gould & Gaucherel (1997)



$$A = \sum_{i=1}^{n} \sum_{j'} p_j (u_{i-1,j} \times u_{i,j}) \bigg/ \sum_{i=1}^{n} s_{i-1} \times s_i \ ,$$

IRS 2B: Adaptive Images

- Bennett 2010, ApJ, 716, 1408
- Begin with image centers (point lens)
- Expand coverage to source boundary
- Radial coord, boundary-sensitive integ.
- Advantage: precision with fewer rays
- Disadvantage: costs more than contour

Linear Limb Darkening

$$S(\theta) = \frac{3F}{(3-u)\pi\theta_*^2}\left[1 - u\left(1 - \sqrt{1 - \frac{\theta^2}{\theta_*^2}}\right)\right]$$

$$S(\theta) = \frac{F}{\pi\theta_*^2}\left[1 - \Gamma\left(1 - \frac{3}{2}\sqrt{1 - \frac{\theta^2}{\theta_*^2}}\right)\right]$$

$$\Gamma = \frac{2u}{3-u}$$

$$\langle r^n \rangle = \frac{\rho^n}{n/2+1}(1 - \alpha_n\Gamma); \qquad \alpha_n = 1 - \frac{(3/2)!(1+n/2)!}{(3/2+n/2)!}$$

$$\langle r^2 \rangle = \frac{\rho^2}{2}\left(1 - \frac{1}{5}\Gamma\right) \qquad \langle r^4 \rangle = \frac{\rho^4}{3}\left(1 - \frac{11}{35}\Gamma\right)$$

Limb Darkening Applications

Point Lens

$$A_{\text{max}} = \langle r^{-1} \rangle = \frac{2}{\rho}\left[1 + \left(\frac{3\pi}{8} - 1\right)\Gamma\right]$$

Hexadecapole

$$A_{\text{finite}} = A_0 \langle r^0 \rangle + A_2 \langle r^2 \rangle + A_4 \langle r^4 \rangle$$

$$= A_0 + \frac{A_2 \rho^2}{2}\left(1 - \frac{1}{5}\Gamma\right) + \frac{A_4 \rho^4}{2}\left(1 - \frac{11}{35}\Gamma\right)$$