Lensing Basics:
III. Basic Theory (continued)

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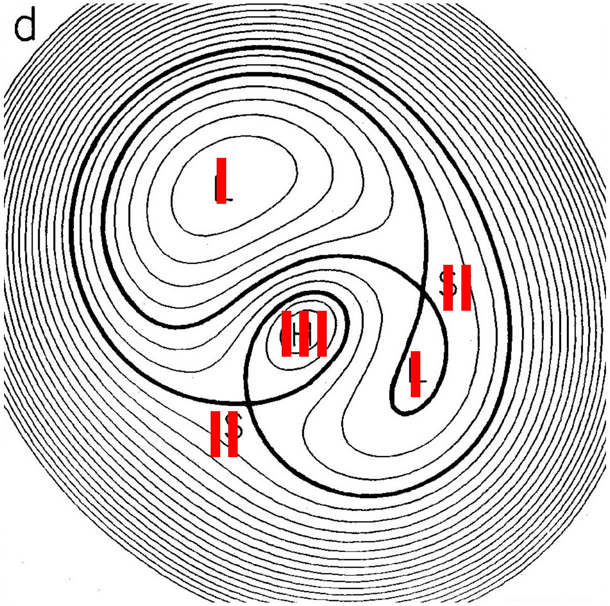
Recap: image classifications

Ordinary (det A $\neq$ 0) images occur at $\nabla\tau(\boldsymbol{\theta};\boldsymbol{\beta}) = 0$
i.e., images are local extrema or saddles of Fermat surface (Fermat’s Principle) for fixed β. Note $\tau_{ij} = A_{ij}$.



[Blandford & Narayan 1986]

Image types:

Type I: minimum of τ
det A > 0; tr A > 0

Type II: saddle point of τ
det A < 0

Type III: maximum of τ
det A > 0; tr A < 0

For mass distributions of finite total mass and that are smooth, there will be at least one Type I image.

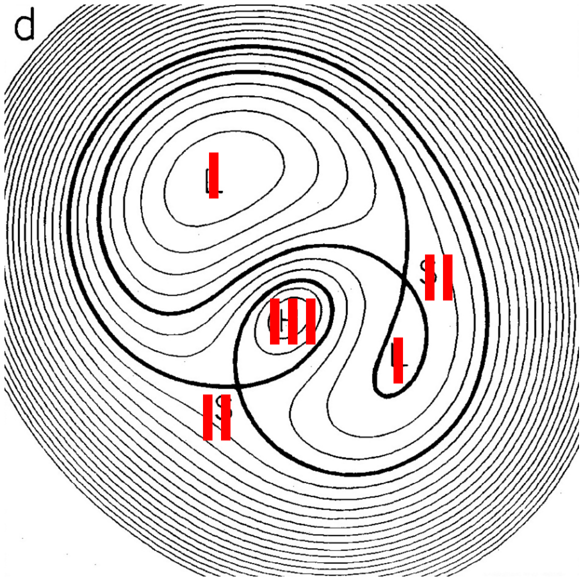
Odd-number theorem

Given smooth $\kappa(\theta)$ that decreases faster than $|\theta|^{-2}$ for $|\theta|\rightarrow\infty$

Then lens has finite total mass, $\alpha(\theta)$ continuous and bounded

Denote n_I, n_{II}, n_{III} = # of Type I, II, III images, respectively

$$n = n_I + n_{II} + n_{III} = \text{total \# of images}$$



[Blandford &Narayan 1986]

For source at position β not on a caustic,

(a) $n_I \geq 1$

(b) $n < \infty$

(c) $n_I + n_{III} = 1 + n_{II}$

(d) For sufficiently large β, $n = n_I = 1$

Therefore,

- total number of images $n = 1 + 2n_{II}$ is odd
- images of positive parity (type I & III) exceed negative parity (type II) by 1 [Burke 1981]
- $n > 1$ if and only if $n_{II} \geq 1$

Magnification theorem

Given smooth $\kappa(\theta)$ that decreases faster than $|\theta|^{-2}$ for $|\theta| \rightarrow \infty$
Then lens has finite total mass, $\alpha(\theta)$ continuous and bounded
Denote n_I, n_{II}, n_{III} = # of Type I, II, III images, respectively

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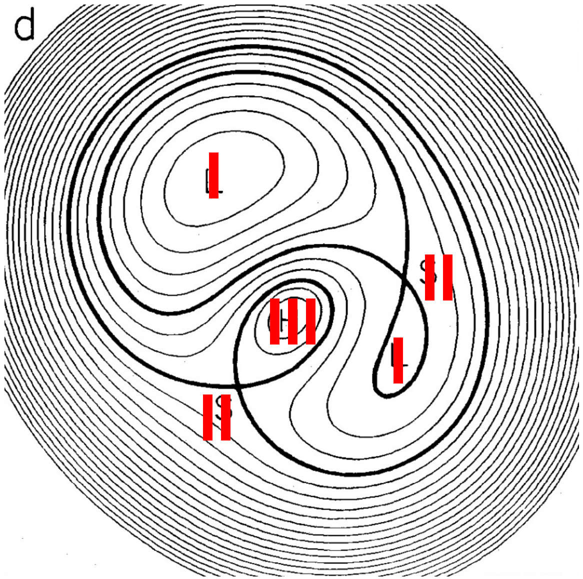


[Blandford &Narayan 1986]

The image of the source which arrives first at the observer is of type I and appears brighter than, or equally bright as the source would appear in the absence of the lens [Schneider 1984]

Conditions for multiple imaging

(a) An isolated transparent lens can produce multiple images if and only if there is a point θ with det $A(\theta) < 0$

(b) A sufficient (but not necessary) condition for possible multiple images is that there exists a point θ such that $\kappa(\theta) > 1$

- Recall $\kappa = \Sigma / \Sigma_{crit}$
- Significance of the critical density Σ_{crit}

Conditions for multiple imaging

Question:

A mass distribution has $\kappa(\theta) < 1$ everywhere. Can it be a strong gravitational lens?

(1) Yes

(2) No

Mass-sheet degeneracy I

Given a lens mass distribution $\kappa(\theta)$ with potential $\psi(\theta)$
Consider the following transformation:

$$\psi_\lambda(\boldsymbol{\theta}) = \frac{\lambda}{2}|\boldsymbol{\theta}|^2 + \underbrace{\boldsymbol{s}\cdot\boldsymbol{\theta}}_{\text{corresponding to constant shift on source plane (unobservable)}} + c + (1-\lambda)\psi(\boldsymbol{\theta})$$

zero point of
lens potential
(unobservable)

Transformed deflection angle ($=\nabla\psi_\lambda$):

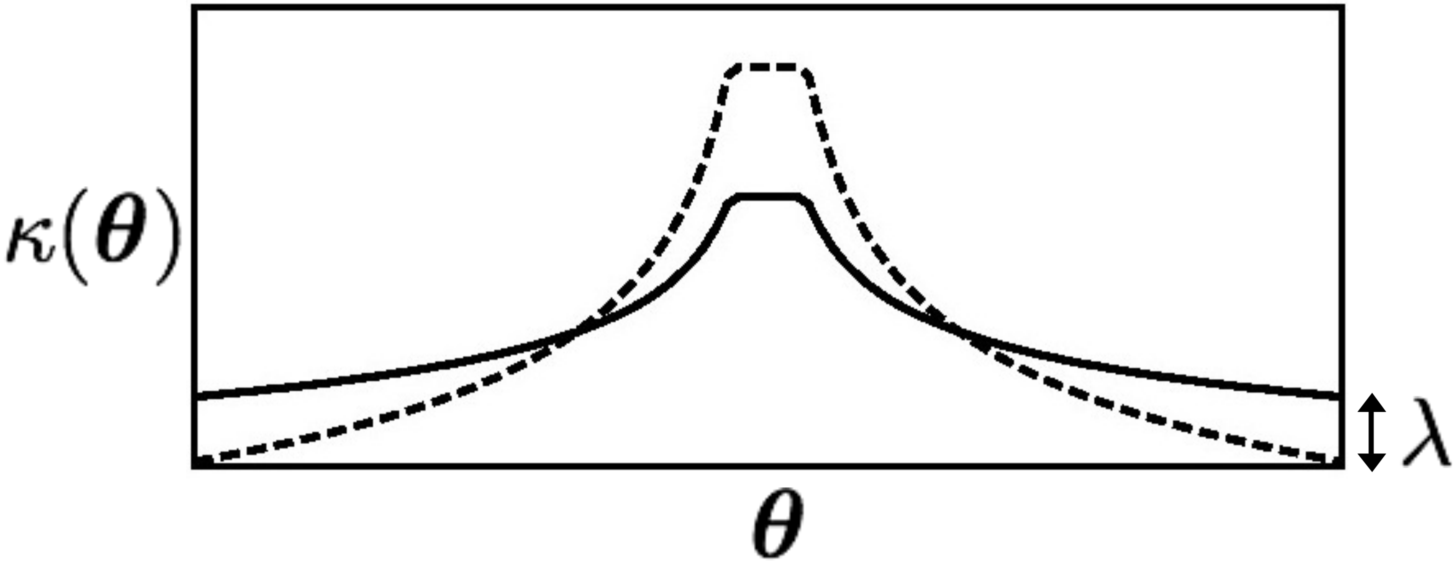
$$\boldsymbol{\alpha}_\lambda(\boldsymbol{\theta}) = \lambda\boldsymbol{\theta} + \boldsymbol{s} + (1-\lambda)\boldsymbol{\alpha}(\boldsymbol{\theta})$$

Transformed convergence ($=\nabla^2\psi_\lambda/2$):

$$\kappa_\lambda(\boldsymbol{\theta}) = \lambda + (1-\lambda)\kappa(\boldsymbol{\theta})$$

Mass-sheet degeneracy II

Last slide: $\kappa_\lambda(\boldsymbol{\theta}) = \lambda + (1-\lambda)\kappa(\boldsymbol{\theta})$



Lens equation:

$$\boldsymbol{\beta}_\lambda = \boldsymbol{\theta} - \boldsymbol{\alpha}_\lambda(\boldsymbol{\theta}) = \boldsymbol{\theta} - \lambda\boldsymbol{\theta} - \boldsymbol{s} - (1-\lambda)\boldsymbol{\alpha}(\boldsymbol{\theta})$$

➡ $$\underbrace{\frac{\boldsymbol{\beta}_\lambda}{1-\lambda} + \frac{\boldsymbol{s}}{1-\lambda}} = \boldsymbol{\theta} - \boldsymbol{\alpha}(\boldsymbol{\theta}) \equiv \boldsymbol{\beta}$$

source scaled and shifted, both unobservable ⇒ degeneracy

Mass-sheet degeneracy III

$$\frac{\boldsymbol{\beta}_\lambda}{1-\lambda} + \frac{\boldsymbol{s}}{1-\lambda} = \boldsymbol{\beta}$$

source scaled and shifted,
both effects unobservable

Magnification

$$\mathcal{A}_\lambda = (1-\lambda)\mathcal{A} \quad \Rightarrow \quad \mu_\lambda = \frac{\mu}{(1-\lambda)^2}$$

Recall

$$\mathcal{A} = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} \gamma_\lambda(\boldsymbol{\theta}) &= (1-\lambda)\gamma(\boldsymbol{\theta}) \\ (1-\kappa_\lambda) &= (1-\lambda)(1-\kappa) \end{aligned}\right\}$$

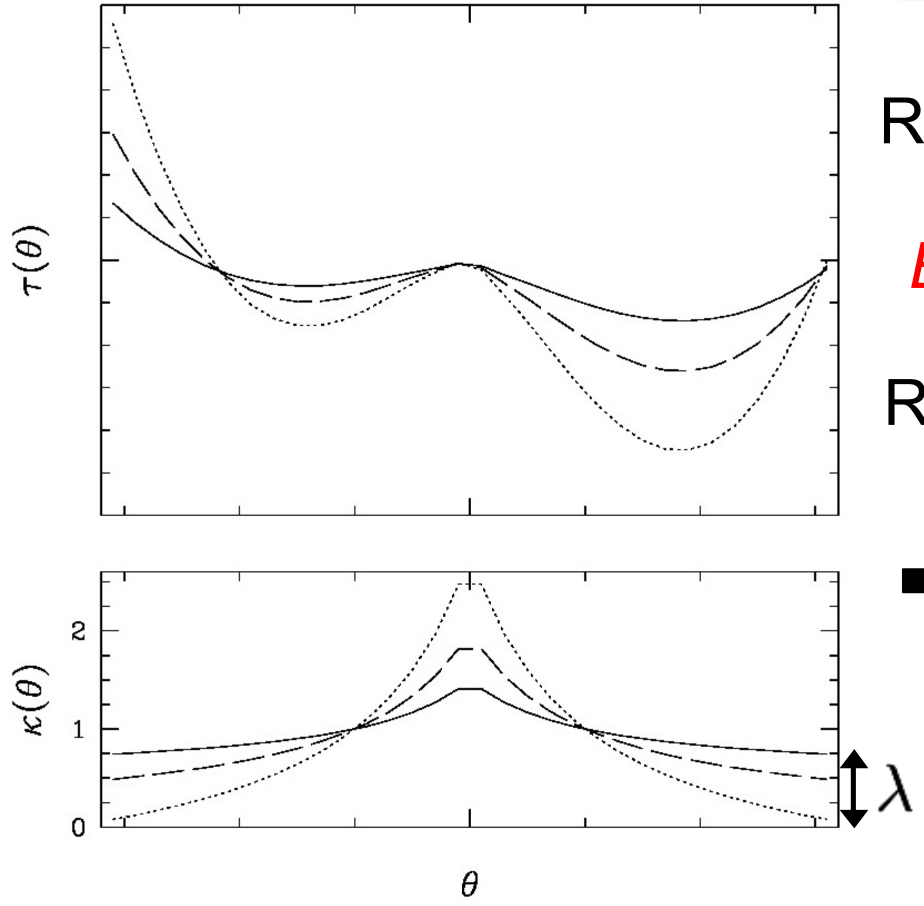
Reduced shear invariant

$$g_\lambda = \frac{\gamma}{1-\kappa} = g$$

Mass-sheet degeneracy IV

Fermat potential: $\tau_\lambda(\boldsymbol{\theta};\boldsymbol{\beta}) = \frac{1}{2}(\boldsymbol{\theta}-\boldsymbol{\beta}_\lambda)^2 - \psi_\lambda(\boldsymbol{\theta})$

$= (1-\lambda)\tau(\boldsymbol{\theta};\boldsymbol{\beta}) + \text{constant}$



[Courbin et al. 2002]

Recall $\kappa_\lambda(\boldsymbol{\theta}) = \lambda + (1-\lambda)\kappa(\boldsymbol{\theta})$

Big impact on cosmography!

Recall $\Delta t(\boldsymbol{\theta};\boldsymbol{\beta}) = \frac{D_{\Delta t}}{c}\Delta\tau(\boldsymbol{\theta};\boldsymbol{\beta})$

➡ For fixed Δt,

model

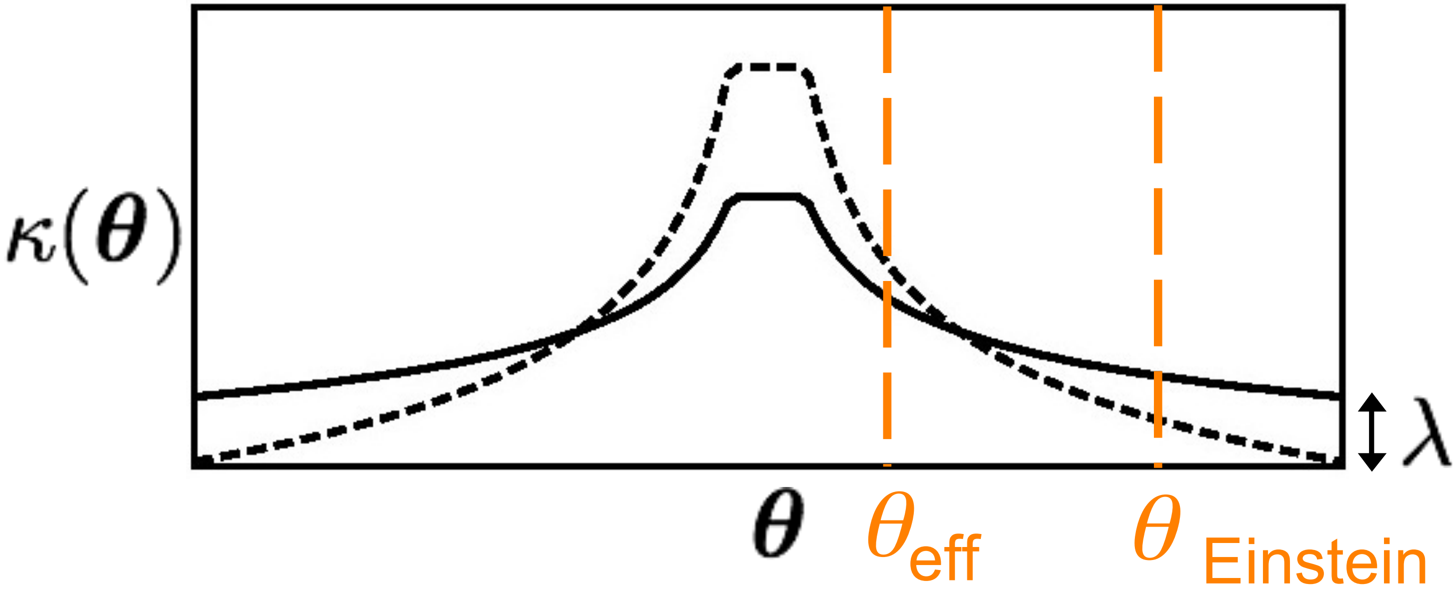
$$D_{\Delta t,\lambda} = \frac{D_{\Delta t}}{1-\lambda}$$

True (including external convergence)

Mass-sheet degeneracy V

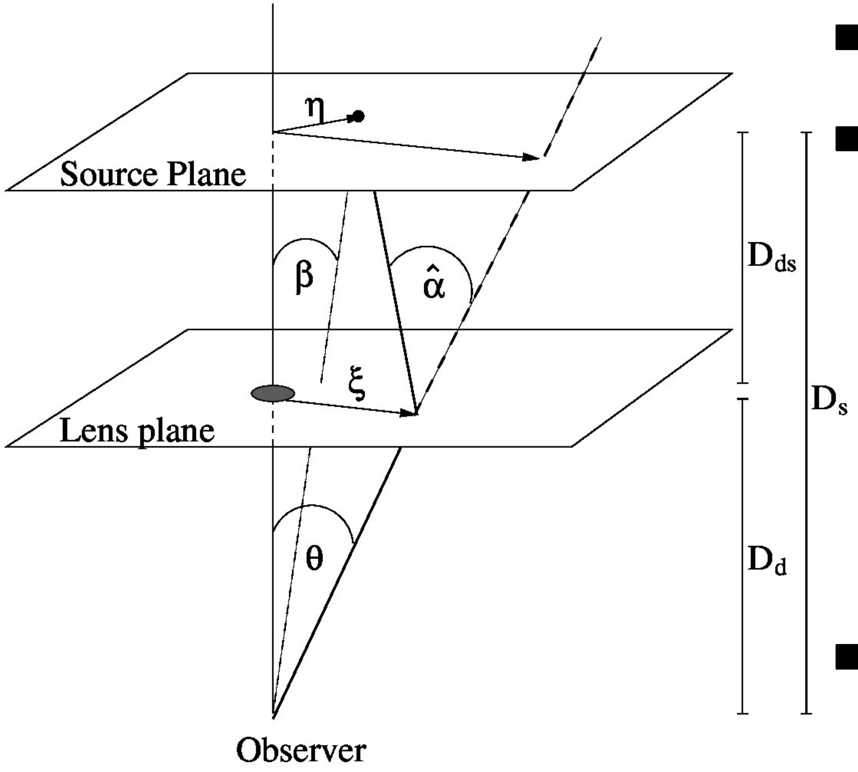
To break the degeneracy:

- need the absolute size or luminosity of source (unpractical)
- stellar kinematics
- study of the lens environment



Simple Lens Models

Recap of Lecture II



[Schneider et al. 2006]

Given a mass distribution $\rho(\boldsymbol{r})$

➡ $\kappa(\theta)$ [convergence]

➡ $\alpha(\theta)$ [scaled deflection angle]

$$\boldsymbol{\alpha}(\boldsymbol{\theta}) = \frac{1}{\pi} \int_{\Re^2} \mathrm{d}^2\theta' \kappa(\boldsymbol{\theta}') \frac{\boldsymbol{\theta} - \boldsymbol{\theta}'}{|\boldsymbol{\theta} - \boldsymbol{\theta}'|^2}$$

lens equation

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \boldsymbol{\alpha}(\boldsymbol{\theta})$$

➡ $\psi(\theta)$ [lens potential]

$$\nabla^2 \psi = 2\kappa$$

➡ $\tau(\boldsymbol{\theta}; \boldsymbol{\beta})$ [Fermat potential]

$$\tau(\boldsymbol{\theta}; \boldsymbol{\beta}) = \frac{1}{2}(\boldsymbol{\theta} - \boldsymbol{\beta})^2 - \psi(\boldsymbol{\theta})$$

Axisymmetric mass distributions I

Axisymmetric mass distribution: $\kappa(\boldsymbol{\theta}) = \kappa(|\boldsymbol{\theta}|)$

Recall $\boldsymbol{\alpha}(\boldsymbol{\theta}) = \frac{1}{\pi} \int_{\Re^2} \mathrm{d}^2\theta' \kappa(\boldsymbol{\theta}') \frac{\boldsymbol{\theta} - \boldsymbol{\theta}'}{|\boldsymbol{\theta} - \boldsymbol{\theta}'|^2}$

For axisymmetric mass distribution:

$$➡ \quad \boldsymbol{\alpha}(\boldsymbol{\theta}) = \frac{2\boldsymbol{\theta}}{|\boldsymbol{\theta}|^2} \int_0^{|\boldsymbol{\theta}|} \mathrm{d}\theta' \, \theta' \kappa(\theta')$$

[Exercise: derive the above equation]

Note: α is collinear with θ.

lens equation, β = θ - α, implies β is also collinear with α

Axisymmetric mass distributions II

Define $\boldsymbol{\beta} = \beta \hat{\mathbf{e}}$ ➡ $\boldsymbol{\theta} = \theta \hat{\mathbf{e}}$

➡ $\boldsymbol{\alpha} = \alpha \hat{\mathbf{e}}$ and $\alpha(\theta) = \frac{2}{\theta} \int_0^\theta \mathrm{d}\theta' \, \theta' \, \kappa(\theta')$

Lens equation reduces to 1-d:

$$\boxed{\beta = \theta - \alpha(\theta)}$$

Note $\alpha(-\theta) = -\alpha(\theta)$

Define mean surface mass density inside circular radius θ:

$\bar{\kappa}(\theta) = \frac{m(\theta)}{\theta^2}$ with the dimensionless mass inside θ

$$m(\theta) = 2 \int_0^\theta \mathrm{d}\theta' \, \theta' \, \kappa(\theta')$$

➡ $\alpha(\theta) = \frac{m(\theta)}{\theta} = \bar{\kappa}(\theta)\theta$

Axisymmetric mass distributions III

Lens equation rewritten as

$$\boldsymbol{\beta} = [1 - \bar{\kappa}(|\boldsymbol{\theta}|)]\,\boldsymbol{\theta}$$

Using $\mathcal{A}(\boldsymbol{\theta}) = \frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{\theta}}$, derive [exercise]

$$\det \mathcal{A} = (1 - \bar{\kappa})(1 + \bar{\kappa} - 2\kappa)$$

critical curves
are defined by
det A = 0

➡

Tangential critical curve:

$$1 - \bar{\kappa}(\theta) = 0$$

Radial critical curve:

$$1 + \bar{\kappa}(\theta) - 2\kappa(\theta) = 0$$

Axisymmetric mass distributions III

Lens equation rewritten as

$$\boldsymbol{\beta} = [1 - \bar{\kappa}(|\boldsymbol{\theta}|)]\, \boldsymbol{\theta}$$

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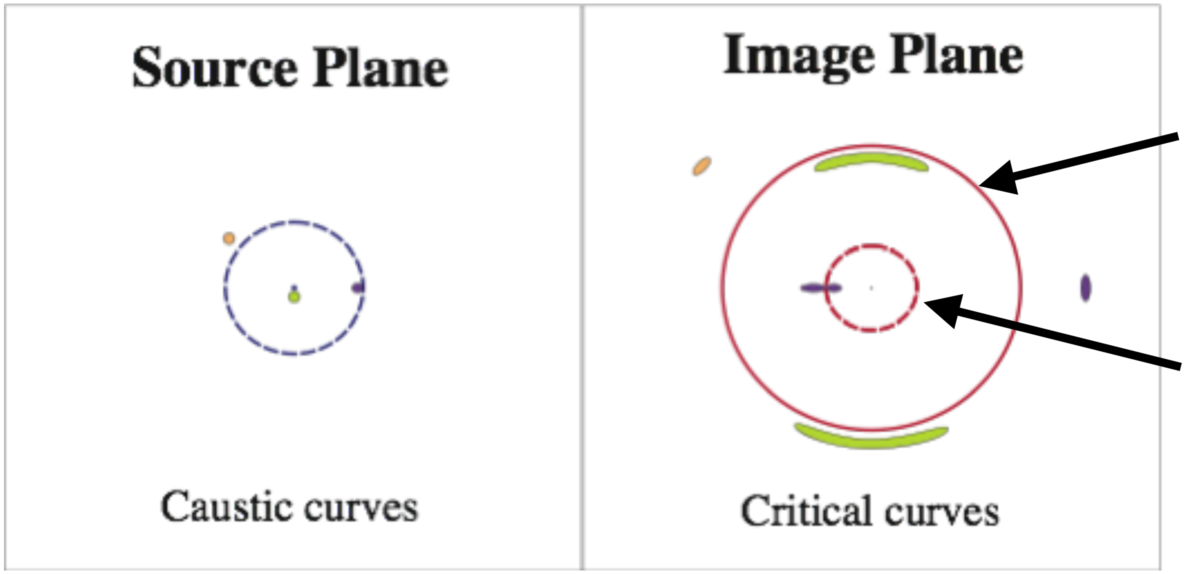


Tangential critical curve:

$$1 - \bar{\kappa}(\theta) = 0$$

Radial critical curve:

$$1 + \bar{\kappa}(\theta) - 2\kappa(\theta) = 0$$

Credit: A. Amara & T. Kitching

Axisymmetric mass distributions IV

Tangential critical curve at radius θ_E has

$$\bar{\kappa}(\theta_{\rm E}) = 1$$

Mass enclosed within θ_E is

$$\begin{aligned} M(\leq \theta_{\rm E}) &= \bar{\kappa}(\theta_{\rm E})\, \pi\theta_{\rm E}^2\, D_{\rm d}^2\, \Sigma_{\rm cr} \\ &= \pi\theta_{\rm E}^2\, D_{\rm d}^2\, \Sigma_{\rm cr} \end{aligned}$$

Rewriting:

$$\begin{aligned} \theta_{\rm E} &= \left(\frac{4GM}{c^2}\frac{D_{\rm ds}}{D_{\rm d}D_{\rm s}}\right)^{1/2} \\ &\approx 0.9'' \left(\frac{M(\leq \theta_{\rm E})}{10^{12}M_\odot}\right)^{1/2} \left(\frac{D_{\rm ds}\, 1{\rm Gpc}}{D_{\rm d}D_{\rm s}}\right)^{1/2} \end{aligned}$$

Mass scale sets radius of tangential critical curve, which is approximately the location of tangential arcs

Singular isothermal sphere I

3 dimensional mass density of SIS: $\rho(r) = \dfrac{\sigma_v^2}{2\pi G r^2}$

Leads to flat rotation curve with rotation velocity $v_{\rm c} = \sqrt{2}\sigma_v$

Surface mass density:

$$\Sigma(\xi) = \int_{-\infty}^{\infty} {\rm d}r_3\, \rho(\sqrt{\xi^2 + r_3^2}) = \frac{\sigma_v^2}{2G}\xi^{-1}$$

Dimensionless surface mass density

$$\kappa(\theta) = \frac{\theta_{\rm E}}{2|\theta|} \quad \text{where} \quad \theta_{\rm E} = 4\pi \left(\frac{\sigma_v}{c}\right)^2 \frac{D_{\rm ds}}{D_{\rm s}}$$

Singular isothermal sphere II

$$\kappa(\theta) = \frac{\theta_{\rm E}}{2|\theta|} \quad \text{where} \quad \theta_{\rm E} = 4\pi \left(\frac{\sigma_v}{c}\right)^2 \frac{D_{\rm ds}}{D_{\rm s}}$$

Properties

$$\bar{\kappa}(\theta) = \frac{\theta_{\rm E}}{|\theta|} \qquad |\gamma|(\theta) = \frac{\theta_{\rm E}}{2|\theta|} \qquad \alpha(\theta) = \theta_{\rm E}\frac{\theta}{|\theta|}$$

[exercise: derive these]

Question:

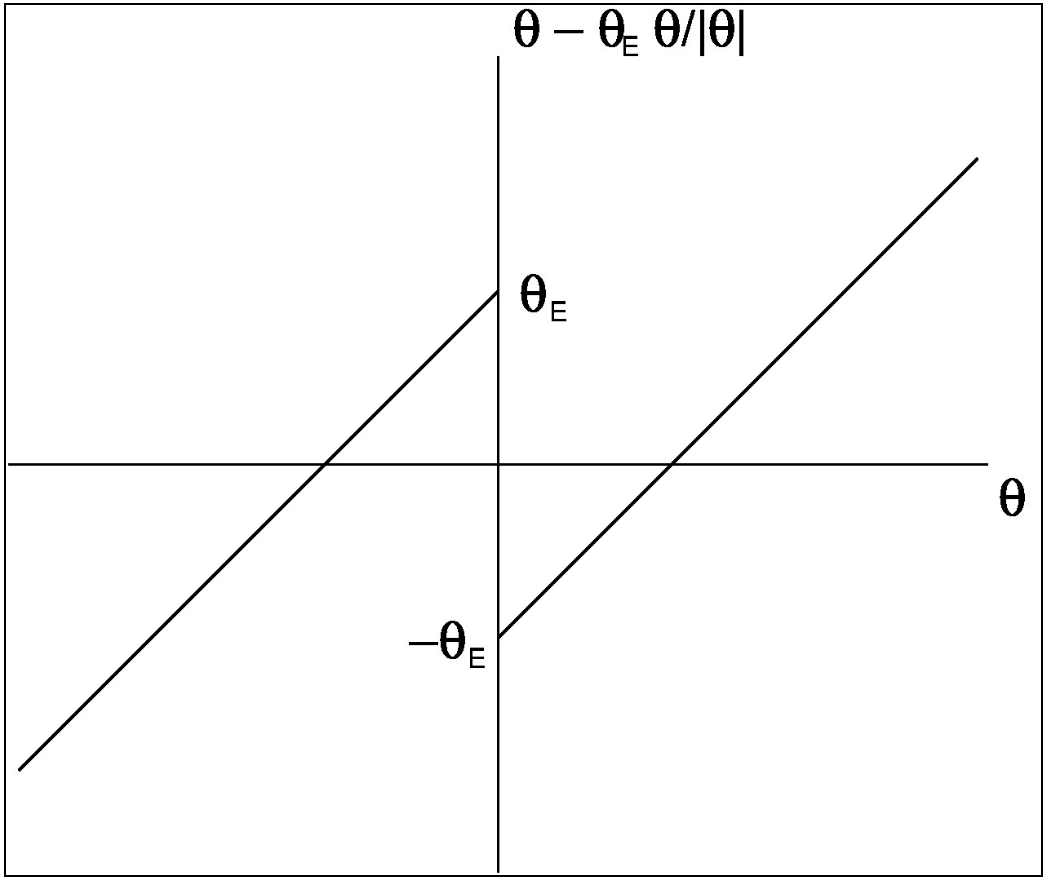
Which of the following is true regarding $\theta_{\rm E} = 4\pi \left(\frac{\sigma_v}{c}\right)^2 \frac{D_{\rm ds}}{D_{\rm s}}$

(1) It corresponds to the radial critical curve
(2) It corresponds to the tangential critical curve
(3) Neither of the above is true

Singular isothermal sphere III

Lens equation with $\alpha(\theta) = \theta_{\rm E} \dfrac{\theta}{|\theta|}$

➡ $\beta = \theta - \theta_{\rm E} \dfrac{\theta}{|\theta|}$



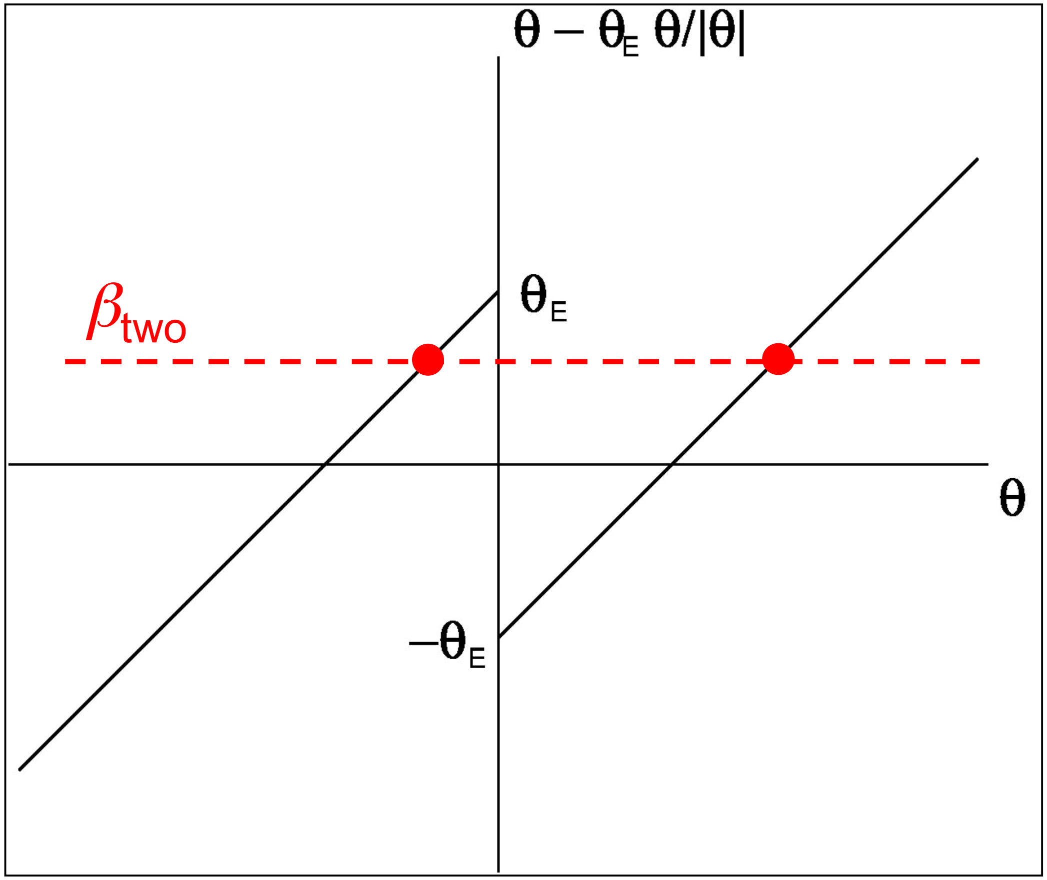
Question:
Which source positions have multiple images?

(1) $\beta > 0$
(2) $\beta > \theta_E$
(3) $-\theta_E < \beta < \theta_E$
(4) $\beta < -\theta_E$
(5) $\beta < 0$
(6) No idea

Singular isothermal sphere III

Lens equation with $\alpha(\theta) = \theta_{\rm E} \dfrac{\theta}{|\theta|}$

$$\Rightarrow \beta = \theta - \theta_{\rm E} \frac{\theta}{|\theta|}$$



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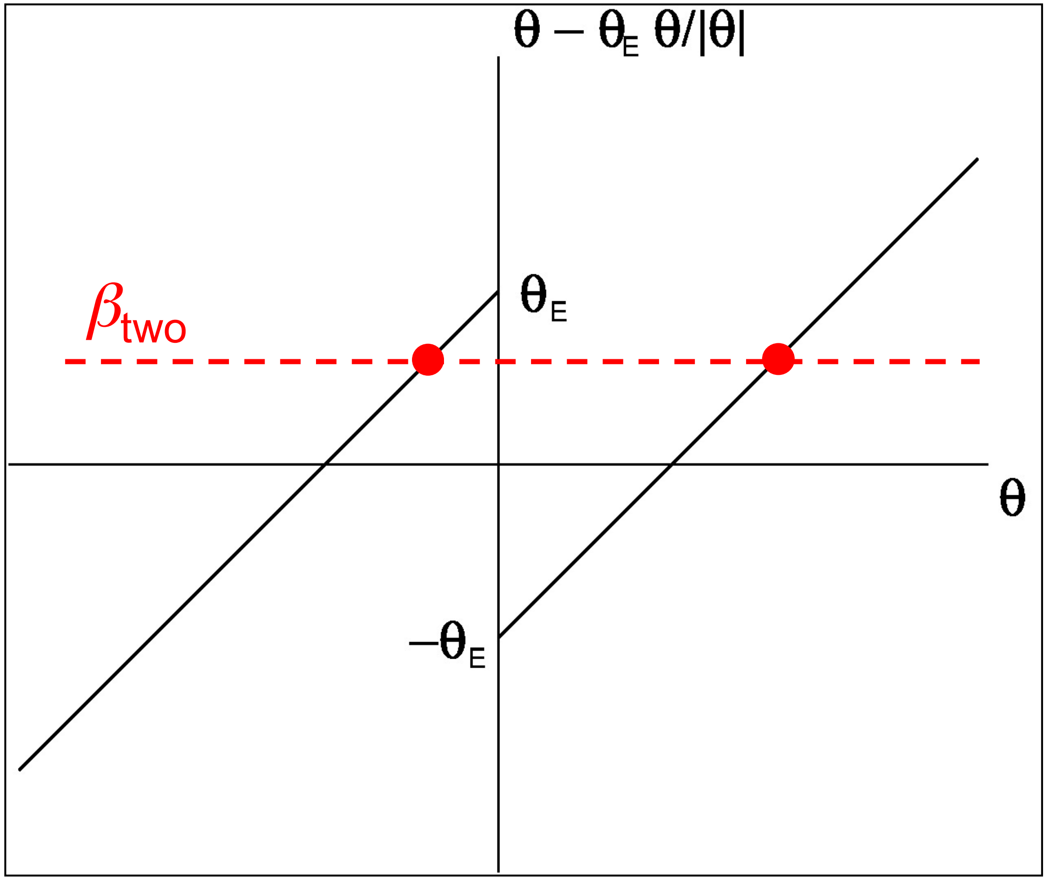
(5) $\beta < 0$
(6) No idea

Singular isothermal sphere IV

Lens equation with $\alpha(\theta) = \theta_{\mathrm{E}} \frac{\theta}{|\theta|}$

$$\Rightarrow \beta = \theta - \theta_{\mathrm{E}} \frac{\theta}{|\theta|}$$



Question:

What is the separation between the two images when $-\theta_E < \beta < \theta_E$?

(1) $\theta_E/2$

(2) θ_E

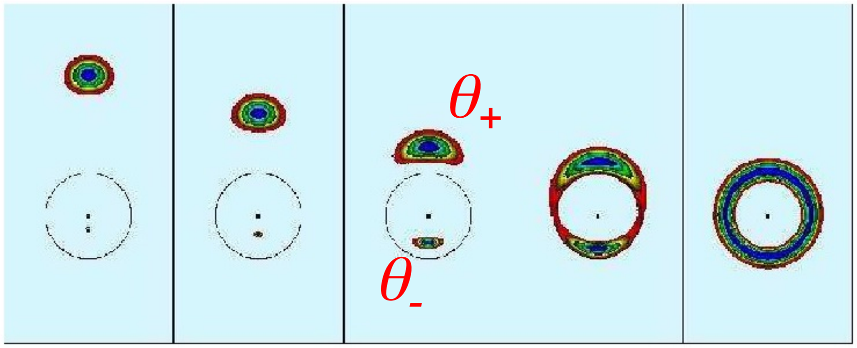
(3) $2\theta_E$

(4) None of the above since the image separation depends on β

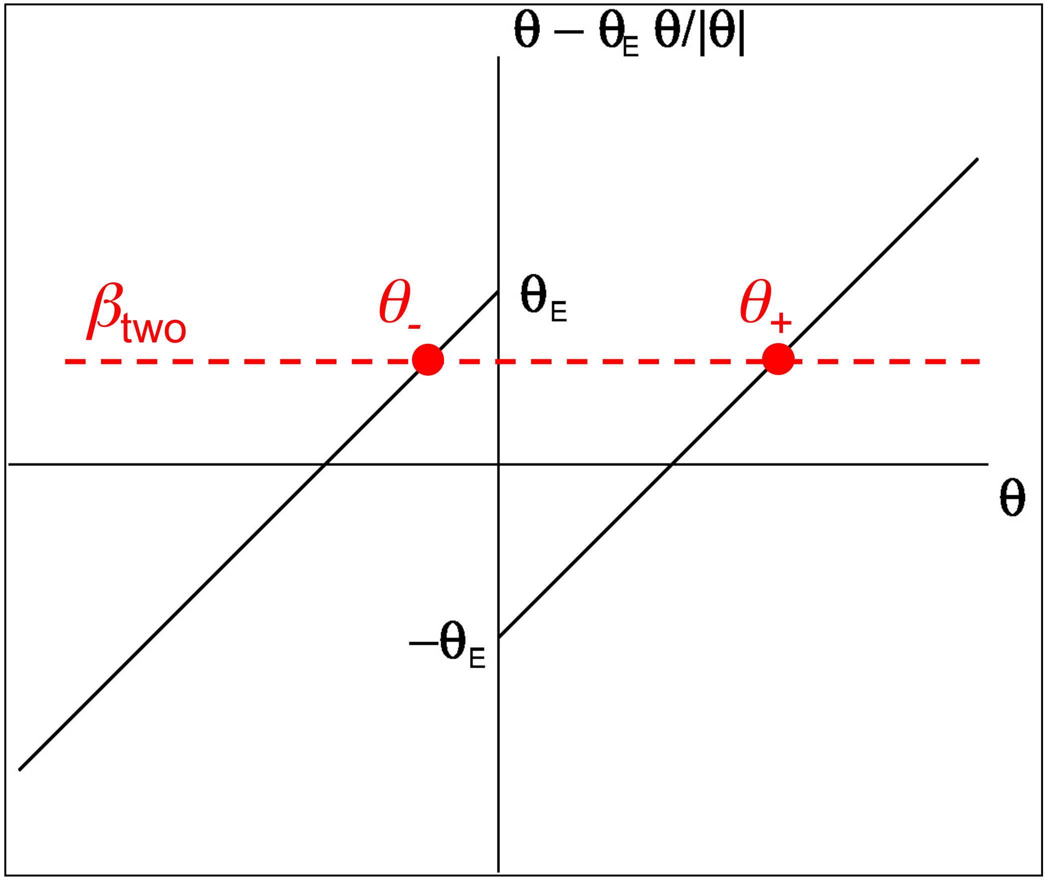
Singular isothermal sphere V

For $-\theta_E < \beta < \theta_E$, the two images are at

$$\theta_+ = \beta + \theta_\mathrm{E}$$
$$\theta_- = \beta - \theta_\mathrm{E}$$



[Wambsganss 1998]



Magnification:

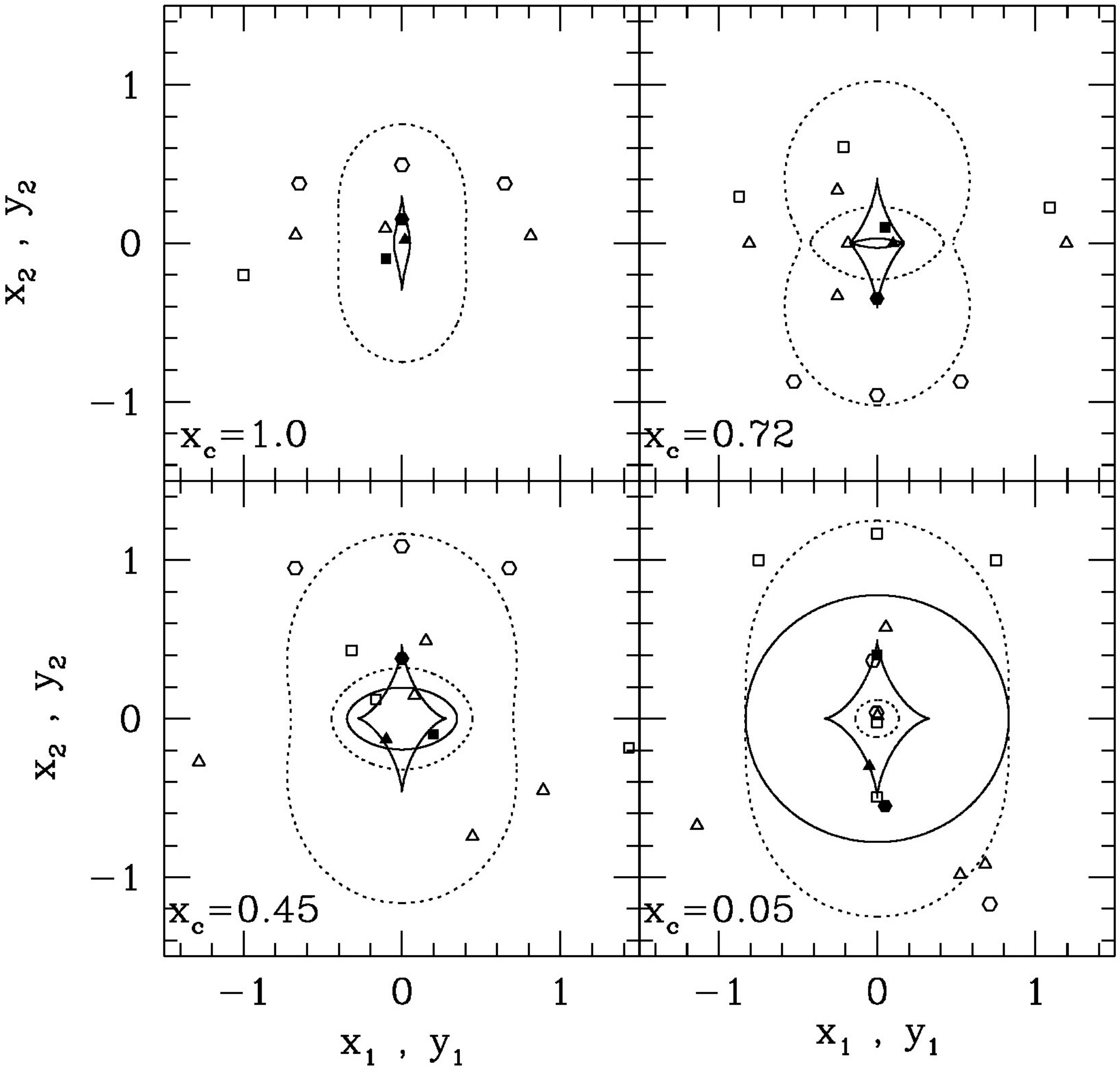
$$\mu = \frac{1}{\det \mathcal{A}} = \frac{|\theta|}{|\theta| - \theta_\mathrm{E}}$$

➡ $\mu_+ > 1$

➡ image at θ_- can be highly demagnified as $\theta_- \rightarrow 0$, or $\beta \rightarrow \theta_E$

Non-singular isothermal sphere with external shear

[Schneider et al. 2006]



Solid curve: caustics

Dashed curve: critical curve

Solid symbols: source position

Open symbols: Image positions

x_c characterizes the core size

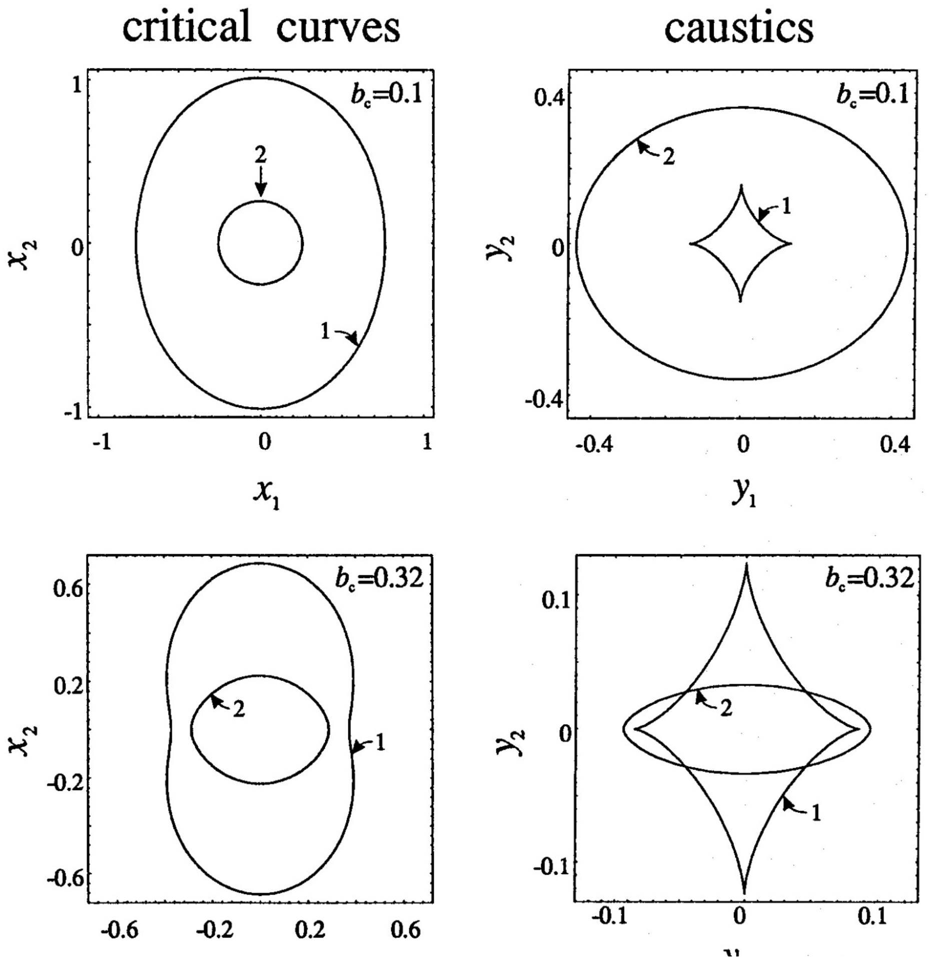
Non-singular isothermal ellipsoid

$$\kappa_{\rm NIE}(\theta_1,\theta_2) = \frac{\theta_{\rm e}}{2\sqrt{\theta_1^2 + f^2\theta_2^2 + b_{\rm c}^2}}$$

strength

core radius

axis ratio of isodensity contour



f=0.8

larger core

isothermal elliptical distributions can produce 4 images (quads)

[Kormann et al. 1994]