Lensing Basics:
II. Basic Theory

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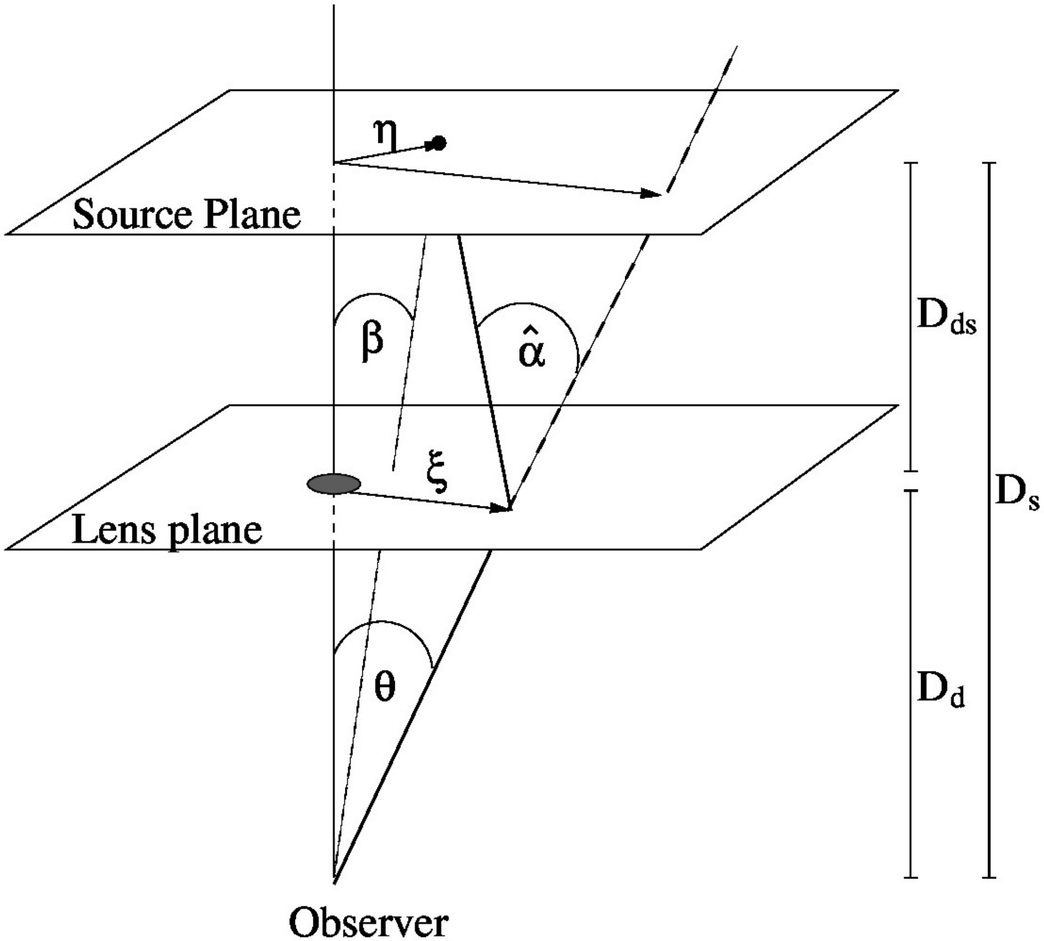
November 5, 2012
@ XXIV Canary Islands Winter School of Astrophysics

Lens equation



[Schneider et al. 2006]

$$\boldsymbol{\eta} = \frac{D_\mathrm{s}}{D_\mathrm{d}}\boldsymbol{\xi} - D_\mathrm{ds}\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi})$$

In terms of angular coord.:

$$\boldsymbol{\eta} = D_\mathrm{s}\boldsymbol{\beta}$$

$$\boldsymbol{\xi} = D_\mathrm{d}\boldsymbol{\theta}$$

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \boldsymbol{\alpha}(\boldsymbol{\theta})$$

where

$$\boldsymbol{\alpha}(\boldsymbol{\theta}) = \frac{D_\mathrm{ds}}{D_\mathrm{s}}\hat{\boldsymbol{\alpha}}(D_\mathrm{d}\boldsymbol{\theta})$$

Deflection angle

Recall from General Relativity:

$$\hat{\boldsymbol{\alpha}} = \frac{4GM}{c^2 \xi}$$

For weak gravitational field and small deflection angles (*geometrically-thin lens*), a light ray with spatial trajectory ($\xi_1(\lambda)$, $\xi_2(\lambda)$, $r_3(\lambda)$) that passes through distribution with 3D density $\rho(\boldsymbol{r})$ will be deflected by

$$\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \frac{4G}{c^2} \int \mathrm{d}^2\xi' \underbrace{\int \mathrm{d}r_3'\, \rho(\xi_1', \xi_2', r_3')}_{\Sigma(\boldsymbol{\xi}')} \frac{\boldsymbol{\xi} - \boldsymbol{\xi}'}{|\boldsymbol{\xi} - \boldsymbol{\xi}'|^2}$$

Scaled deflection angle

recall $\boldsymbol{\alpha}(\boldsymbol{\theta}) = \frac{D_{\rm ds}}{D_{\rm s}} \hat{\boldsymbol{\alpha}}(D_{\rm d}\boldsymbol{\theta})$

$$\boldsymbol{\alpha}(\boldsymbol{\theta}) = \frac{1}{\pi} \int_{\Re^2} {\rm d}^2\theta' \kappa(\boldsymbol{\theta}') \frac{\boldsymbol{\theta} - \boldsymbol{\theta}'}{|\boldsymbol{\theta} - \boldsymbol{\theta}'|^2}$$

where κ is the dimensionless surface mass density (a.k.a. convergence)

$$\kappa(\boldsymbol{\theta}) = \frac{\Sigma(D_{\rm d}\boldsymbol{\theta})}{\Sigma_{\rm cr}}$$

and Σ_{cr} is the critical surface mass density

$$\Sigma_{\rm cr} = \frac{c^2}{4\pi G} \frac{D_{\rm s}}{D_{\rm d}\, D_{\rm ds}}$$

Lens potential

$$\boldsymbol{\alpha}(\boldsymbol{\theta}) = \frac{1}{\pi} \int_{\Re^2} \mathrm{d}^2\theta' \kappa(\boldsymbol{\theta}') \frac{\boldsymbol{\theta} - \boldsymbol{\theta}'}{|\boldsymbol{\theta} - \boldsymbol{\theta}'|^2}$$

Using $\nabla \ln |\boldsymbol{\theta}| = \boldsymbol{\theta}/|\boldsymbol{\theta}^2|$

$$\boldsymbol{\alpha}(\boldsymbol{\theta}) = \nabla \psi(\boldsymbol{\theta})$$

where the lens potential is

$$\psi(\boldsymbol{\theta}) = \frac{1}{\pi} \int_{\Re^2} \mathrm{d}^2\theta' \kappa(\boldsymbol{\theta}') \ln |\boldsymbol{\theta} - \boldsymbol{\theta}'|$$

that satisfies the Poisson equation:

$$\nabla^2 \psi = 2\kappa$$

Fermat potential

Define scalar function known as the Fermat potential

$$\tau(\boldsymbol{\theta};\boldsymbol{\beta}) = \frac{1}{2}(\boldsymbol{\theta}-\boldsymbol{\beta})^2 - \psi(\boldsymbol{\theta})$$

It is a function of θ with β as a parameter

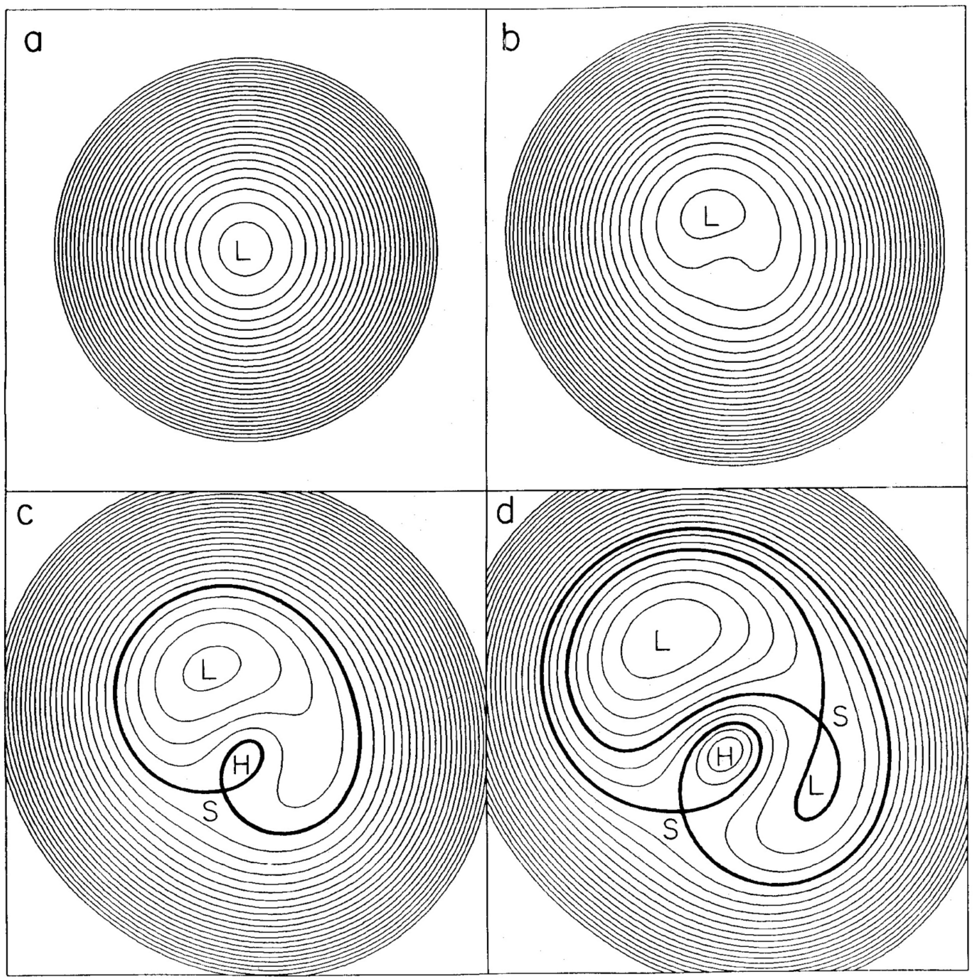
Note:

$$\nabla\tau(\boldsymbol{\theta};\boldsymbol{\beta}) = 0$$

yields the lens equation $\boldsymbol{\beta} = \boldsymbol{\theta} - \boldsymbol{\alpha}(\boldsymbol{\theta})$

Fermat's Principle

Fermat potential $\tau \propto$ excess time delay t



[Blandford & Narayan 1986]

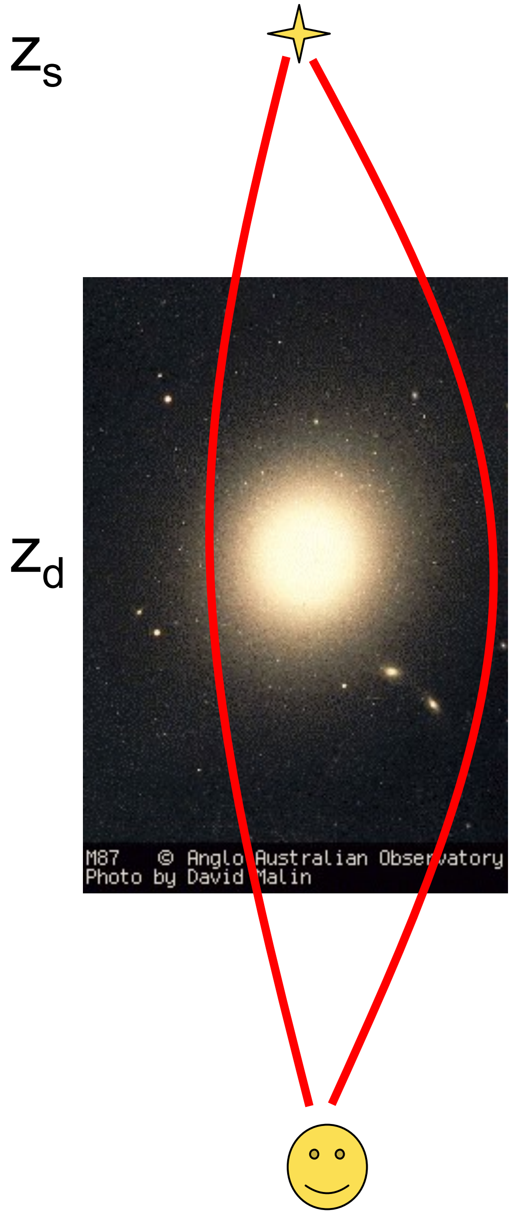
Fermat's Principle:

rays of light traverse the path of stationary optical length with respect to variations of the path

i.e.,

$$\nabla t = \nabla \tau = 0$$

Brief recap



Given a mass distribution $\rho(\boldsymbol{r})$, redshifts z_d, z_s

➡ $\kappa(\boldsymbol{\theta})$ [convergence]

➡ $\boldsymbol{\alpha}(\boldsymbol{\theta})$ [scaled deflection angle]

$$\boldsymbol{\alpha}(\boldsymbol{\theta}) = \frac{1}{\pi}\int_{\Re^2} \mathrm{d}^2\theta' \kappa(\boldsymbol{\theta}') \frac{\boldsymbol{\theta}-\boldsymbol{\theta}'}{|\boldsymbol{\theta}-\boldsymbol{\theta}'|^2}$$

lens equation (governs light paths)

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \boldsymbol{\alpha}(\boldsymbol{\theta})$$

Lenses are your dark matter goggles

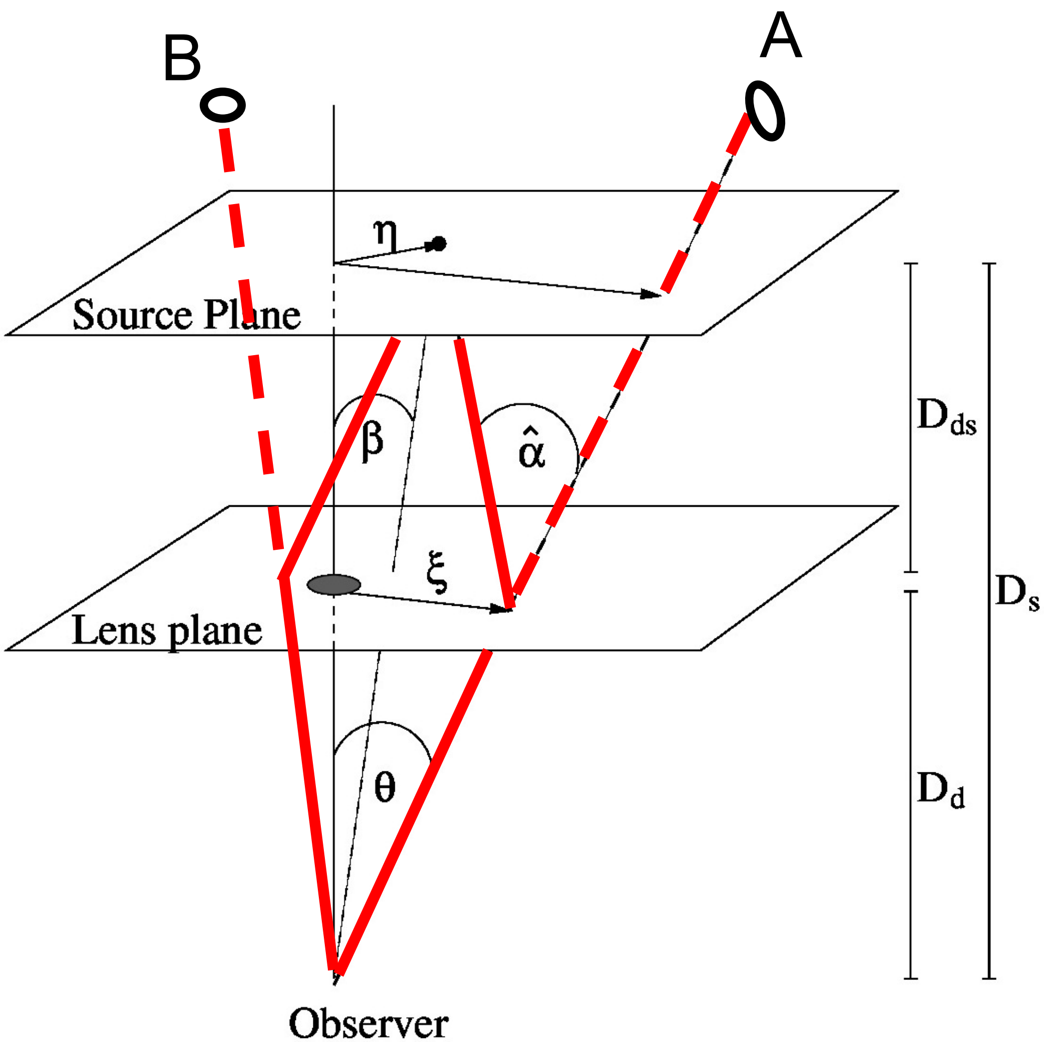
➡ $\psi(\boldsymbol{\theta})$ [lens potential]

$$\nabla^2\psi = 2\kappa$$

➡ $\tau(\boldsymbol{\theta};\boldsymbol{\beta})$ [Fermat potential]

$$\tau(\boldsymbol{\theta};\boldsymbol{\beta}) = \frac{1}{2}(\boldsymbol{\theta}-\boldsymbol{\beta})^2 - \psi(\boldsymbol{\theta})$$

Time delay



[Schneider et al. 2006]

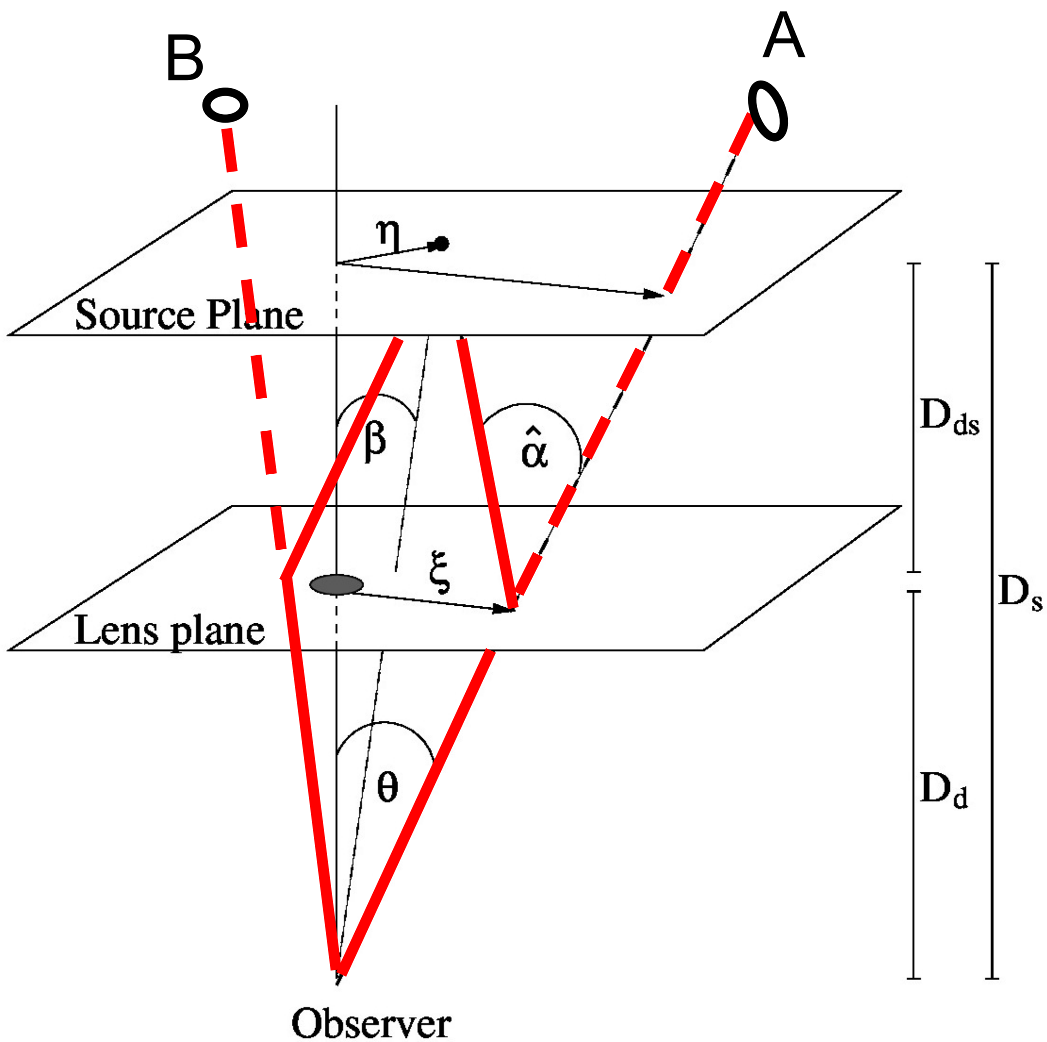
Recall Fermat potential

$$\tau(\boldsymbol{\theta};\boldsymbol{\beta}) = \frac{1}{2}(\boldsymbol{\theta}-\boldsymbol{\beta})^2 - \psi(\boldsymbol{\theta})$$

Excess time delay relative to the case of no lensing is

$$t(\boldsymbol{\theta};\boldsymbol{\beta}) = \frac{D_\mathrm{d} D_\mathrm{s}}{c\,D_\mathrm{ds}}(1+z_\mathrm{d})\tau(\boldsymbol{\theta};\boldsymbol{\beta})$$

Time delay



[Schneider et al. 2006]

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$$\equiv D_{\Delta t} \propto H_0^{-1}$$

depends on cosmology

depends on lens mass distrib.

Cosmology Probe

Magnification

Lensing conserves surface brightness
Flux F = surface brightness x solid angle
Magnification = $F_{observed} / F_{intrinsic} = d\Omega_{observed} / d\Omega_{intrinsic}$

Define Jacobian matrix:

$$\mathcal{A}(\boldsymbol{\theta}) = \frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{\theta}} \qquad \text{with} \quad \mathcal{A}_{ij} = \frac{\partial \beta_i}{\partial \theta_j}$$

source plane (→ β_i)
lens plane (→ θ_j)

Magnification factor is

$$\mu(\boldsymbol{\theta}) = \frac{1}{\det \mathcal{A}(\boldsymbol{\theta})}$$

- $\mu>0$: positive parity
- $\mu<0$: negative parity (mirror image of source)
- det A = 0 : critical points/curves

Image distortion I

Rewrite Jacobian matrix:

$$\mathcal{A}(\boldsymbol{\theta}) = \frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{\theta}} = \left(\delta_{ij} - \frac{\partial^2 \psi(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j}\right) = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}$$

where γ_1 and γ_2 are the two components of shear

$$\gamma \equiv \gamma_1 + \mathrm{i}\gamma_2 = |\gamma| \mathrm{e}^{2\mathrm{i}\varphi}$$

$$\gamma_1 = \frac{1}{2}(\psi_{,11} - \psi_{,22}) \qquad \gamma_2 = \psi_{,12}$$

Magnification in terms of κ and γ is:

$$\mu = \frac{1}{\det \mathcal{A}} = \frac{1}{(1-\kappa)^2 - |\gamma|^2}$$

Image distortion II

Surface brightness conservation:

$$I(\boldsymbol{\theta}) = I^{(\mathrm{s})}[\boldsymbol{\beta}(\boldsymbol{\theta})]$$

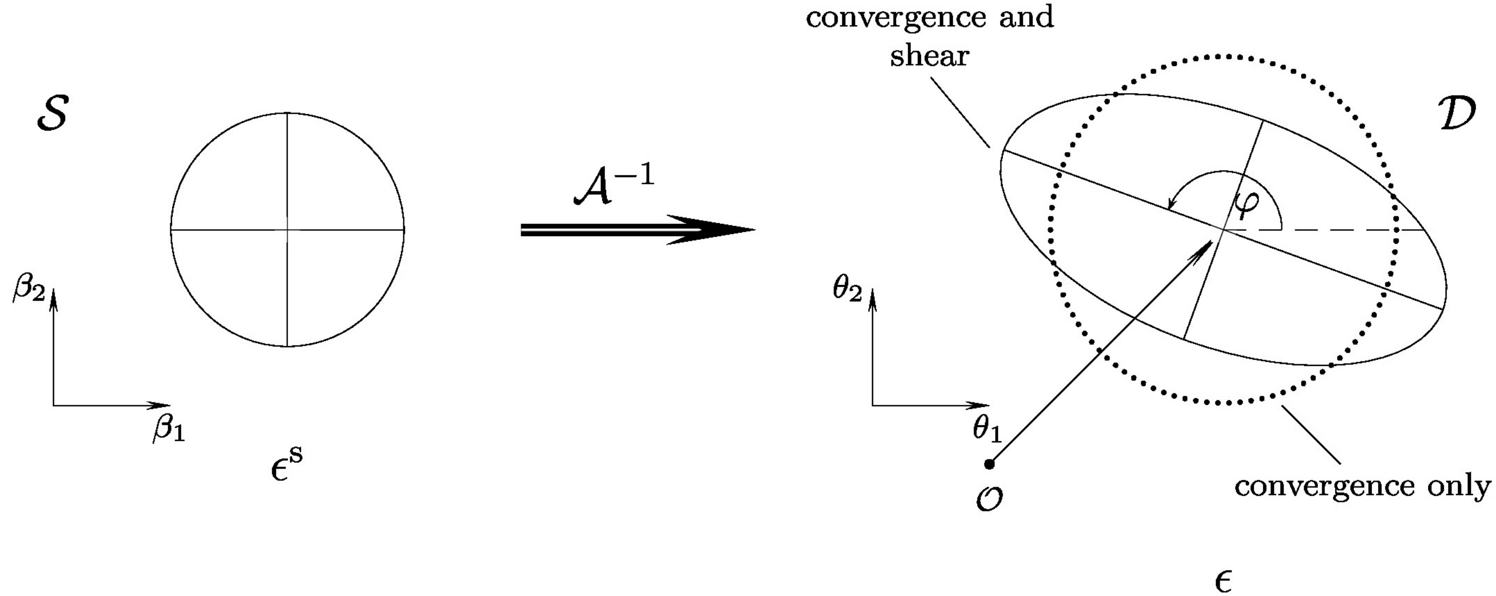
To visualize distortion, consider locally linearized lens eq.:

$$\boldsymbol{\beta} - \boldsymbol{\beta}_0 = \mathcal{A}(\boldsymbol{\theta}_0) \cdot (\boldsymbol{\theta} - \boldsymbol{\theta}_0)$$

Question: for an infinitesimally small circular source, what would the shape of its lensed image be?

(1) Circular
(2) Elliptical
(3) Boxy
(4) Irregular
(5) None of the above

Image distortion III



Credit: M. Bradac

The lensed image of a small circular source with radius R is an ellipse

Major axis:

$$\frac{R}{1-\kappa-|\gamma|}=\frac{R}{(1-\kappa)(1-|g|)}$$

;

Minor axis:

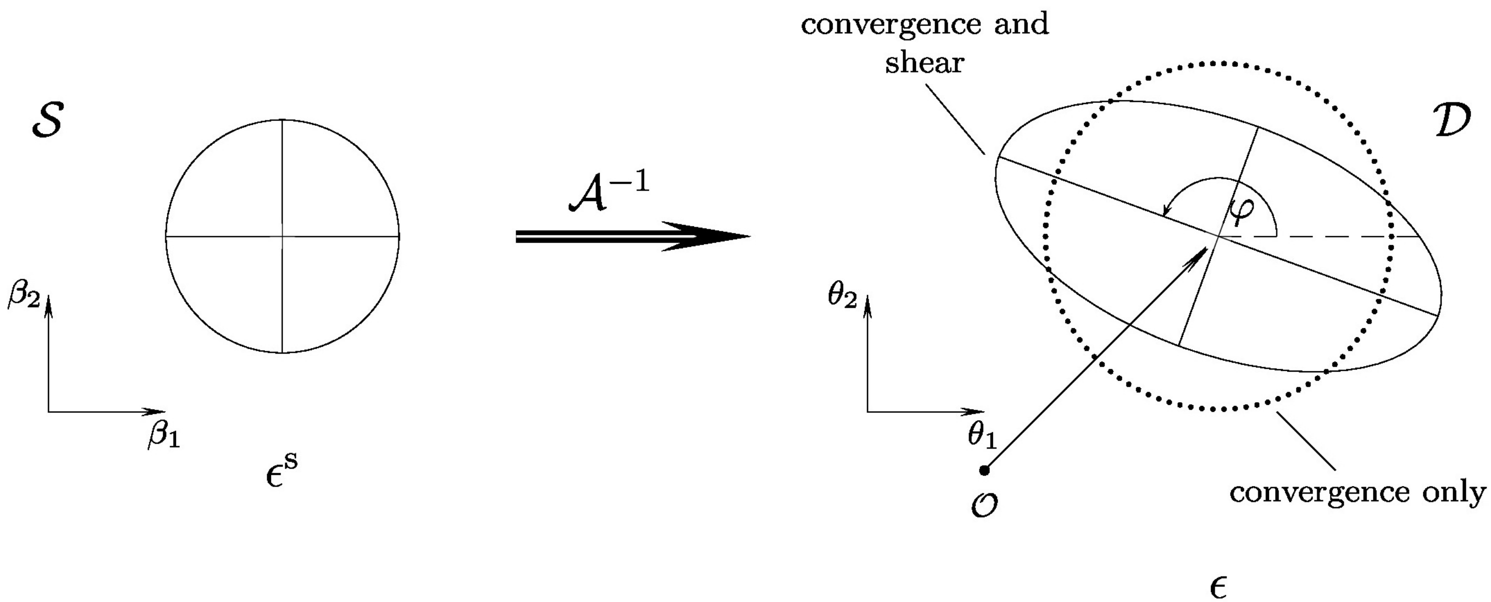
$$\frac{R}{1-\kappa+|\gamma|}=\frac{R}{(1-\kappa)(1+|g|)}$$

reduced shear $g(\boldsymbol{\theta})=\frac{\gamma(\boldsymbol{\theta})}{[1-\kappa(\boldsymbol{\theta})]}$

Angle of major axis from θ_1 the same as the shear angle φ

[Exercise: show these properties. Hint: try $\boldsymbol{\beta}(\lambda)=\boldsymbol{\beta}_0+R(\cos\lambda,\sin\lambda)$]

Image distortion IV



Credit: M. Bradac

Axis ratio of ellipse:

$$\frac{b}{a} = \frac{R}{(1-\kappa)(1+|g|)} \Big/ \frac{R}{(1-\kappa)(1-|g|)} = \frac{1-|g|}{1+|g|}$$

➡ shapes of lensed images yield estimate of reduced shear

$$|g| = \frac{1-b/a}{1+b/a}$$

BUT sources are not intrinsically round…

➡ average over many sources, and assume intrinsic ellipticities are *randomly oriented*

Shear and convergence

Recall

$$\gamma_1 = \frac{1}{2}(\psi_{,11} - \psi_{,22}) \qquad \gamma_2 = \psi_{,12}$$

$$\psi(\boldsymbol{\theta}) = \frac{1}{\pi}\int_{\Re^2} \mathrm{d}^2\theta' \kappa(\boldsymbol{\theta'}) \ln|\boldsymbol{\theta} - \boldsymbol{\theta'}|$$

Shear can therefore be written as

$$\gamma(\boldsymbol{\theta}) = \frac{1}{\pi}\int_{\Re^2} \mathrm{d}^2\theta' \, \mathcal{D}(\boldsymbol{\theta} - \boldsymbol{\theta'}) \, \kappa(\boldsymbol{\theta'})$$

$$\mathcal{D}(\boldsymbol{\theta}) \equiv -\frac{\theta_1^2 - \theta_2^2 + 2\mathrm{i}\theta_1\theta_2}{|\boldsymbol{\theta}|^4} = \frac{-1}{(\theta_1 - \mathrm{i}\theta_2)^2}$$

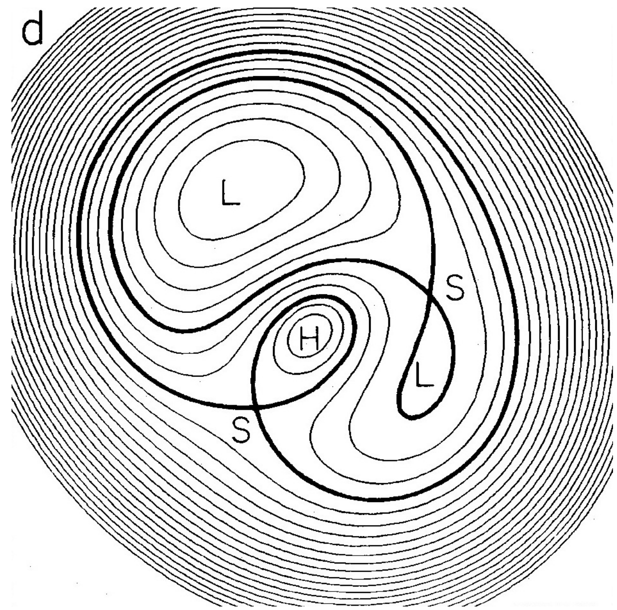
Inverting this:

$$\kappa(\boldsymbol{\theta}) - \kappa_0 = \frac{1}{\pi}\int_{\Re^2} \mathrm{d}^2\theta' \, \mathcal{R}e\left[\mathcal{D}^*(\boldsymbol{\theta} - \boldsymbol{\theta'})\, \gamma(\boldsymbol{\theta'})\right]$$

➡ mass reconstruction from weak lensing

Classification of ordinary images

Ordinary (det A ≠ 0) images occur at $\nabla\tau(\boldsymbol{\theta};\boldsymbol{\beta}) = 0$
i.e., images are local extrema or saddles of Fermat surface (Fermat's Principle) for fixed $\boldsymbol{\beta}$. Note $\tau_{ij} = A_{ij}$.



[Blandford & Narayan 1986]

<u>Image types:</u>

Type I: minimum of τ
det A > 0; tr A > 0

Type II: saddle point of τ
det A < 0

Type III: maximum of τ
det A > 0; tr A < 0

For mass distributions of finite total mass and that are smooth, there will be at least one Type I image.

Multiple Images

Question:

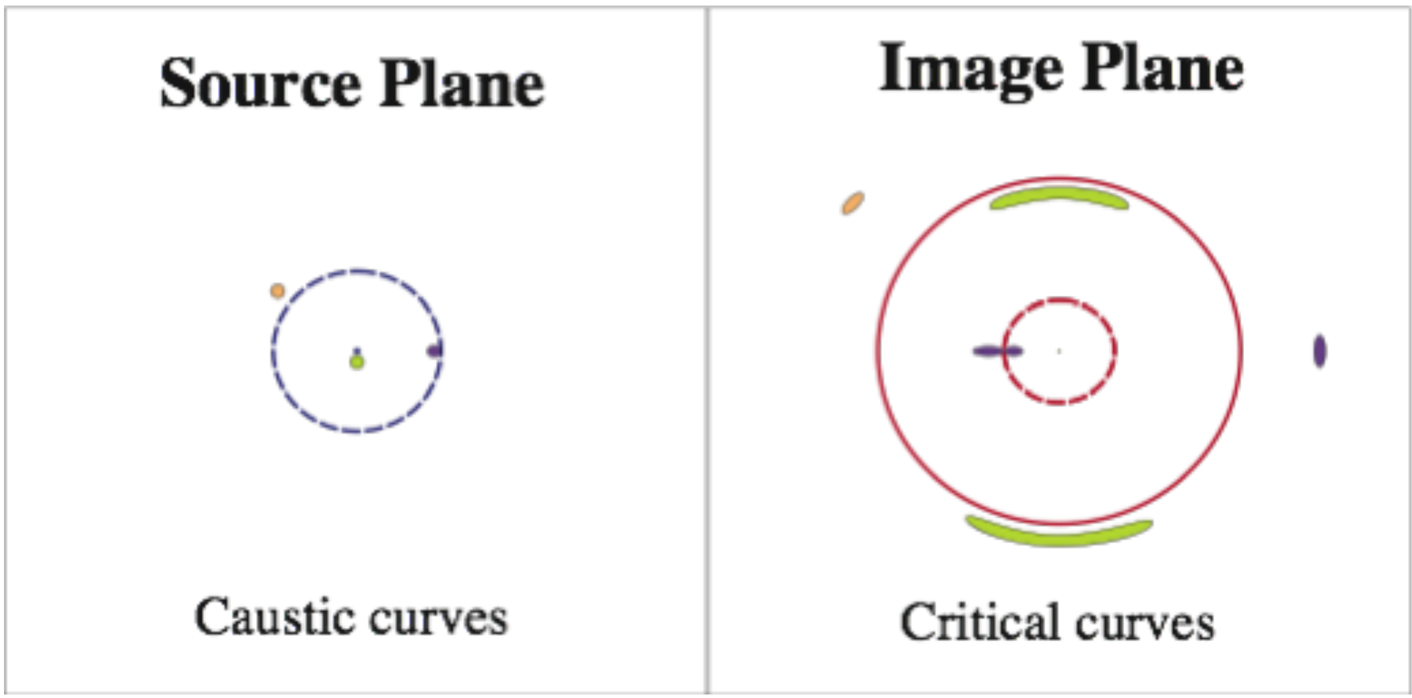
Given a lens mass distribution that is smooth with det A > 0 everywhere, what is the highest number of lensed images of a background source?

1) 1
2) 2
3) 3
4) 4
5) Unlimited, depending on the precise form of the lens mass distribution

Critical curves and caustics I

det A = 0: critical curves on image plane θ
corresponds to caustics on source plane β

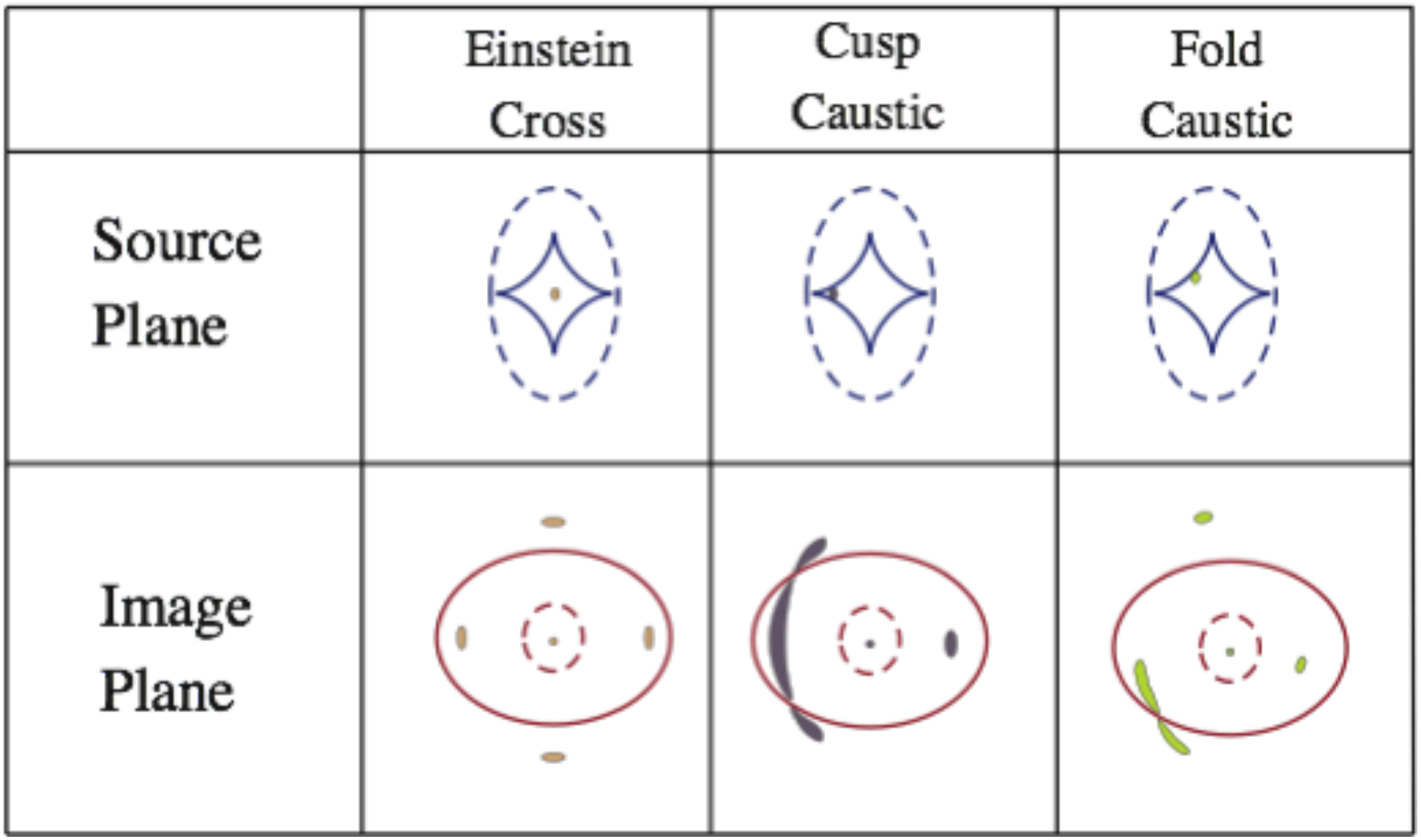
Example: non-singular isothermal sphere lens $\rho(r) = \dfrac{\rho_0}{r_c^2 + r^2}$



Credit: A. Amara & T. Kitching

Critical curves and caustics II

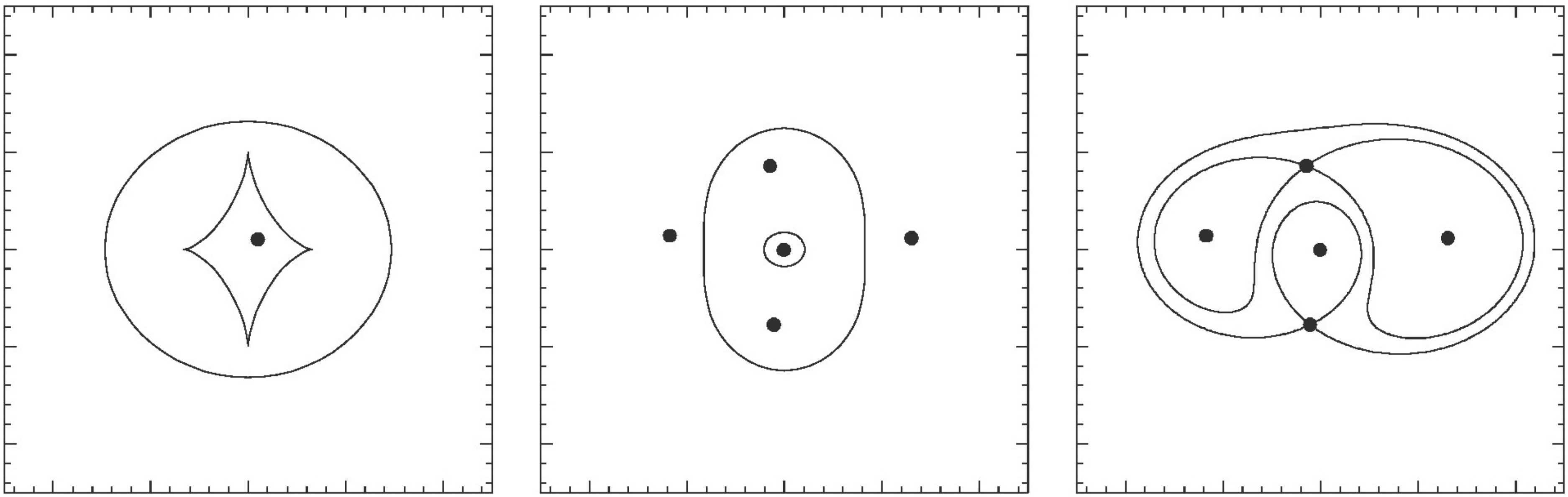
Example: non-singular isothermal ellipsoid lens



Credit: A. Amara & T. Kitching

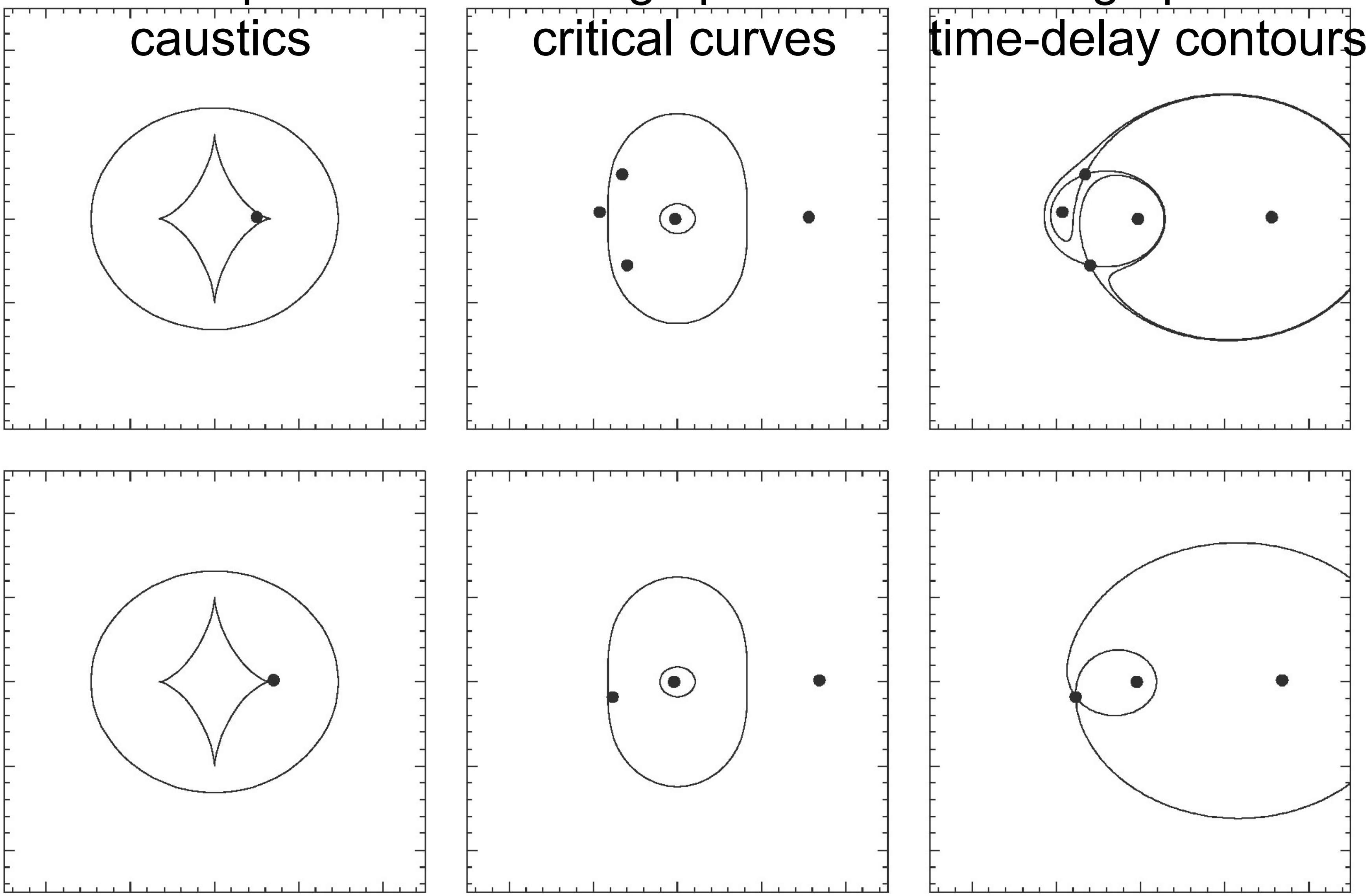
Critical curves and caustics III

source plane caustics | image plane critical curves | image plane time-delay contours



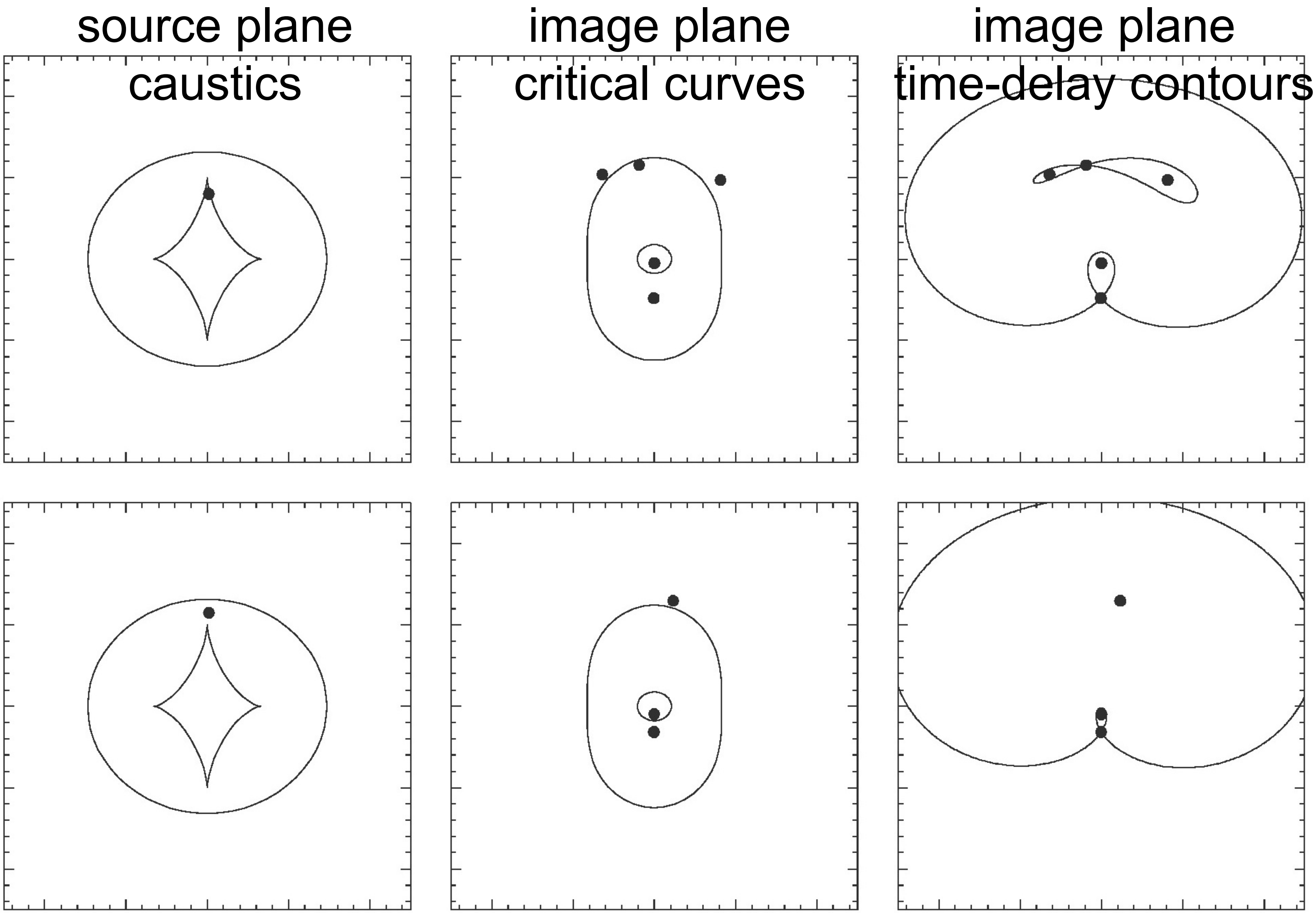
[Courbin et al. 2002]

Critical curves and caustics IV



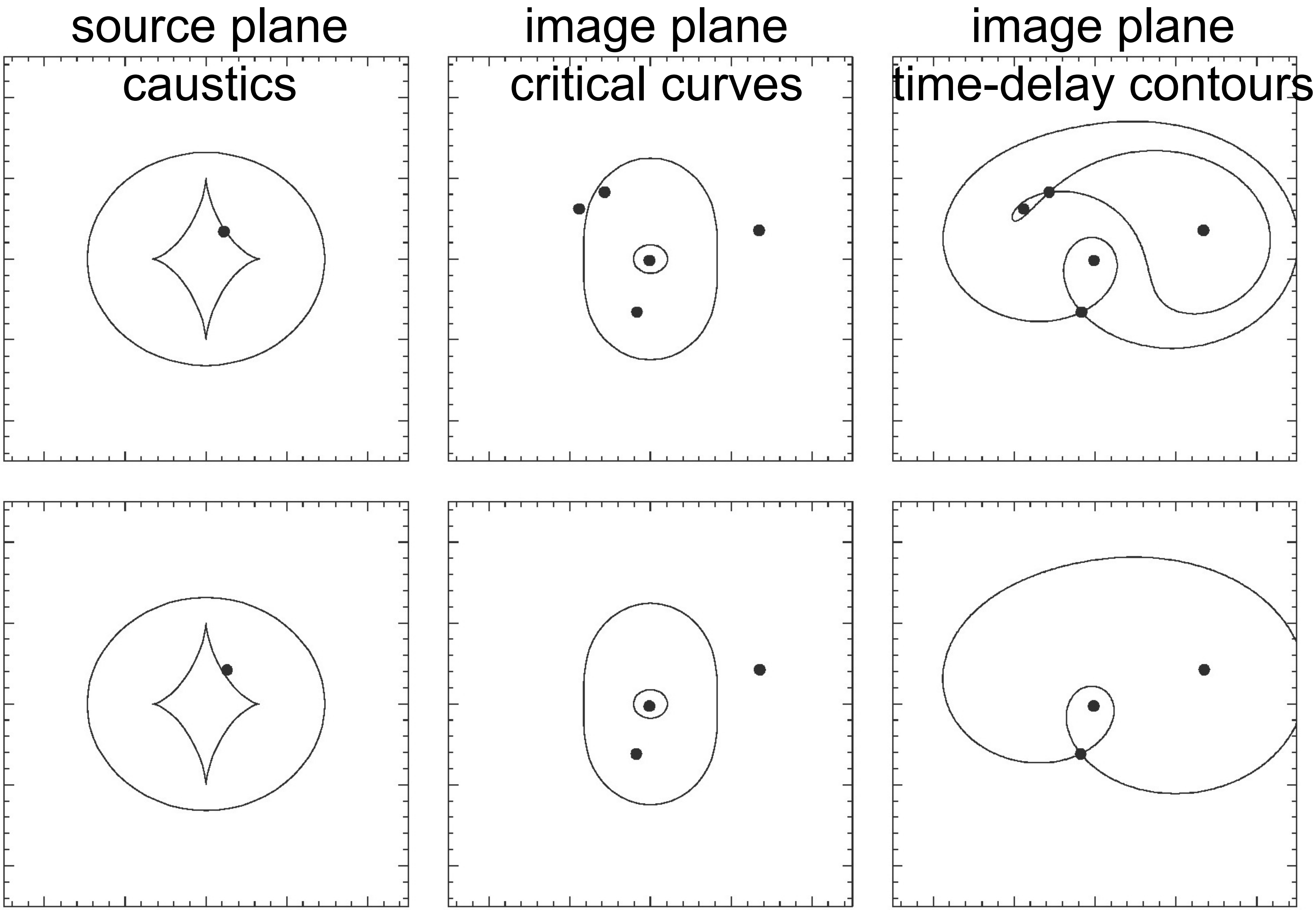
[Courbin et al. 2002]

Critical curves and caustics V



[Courbin et al. 2002]

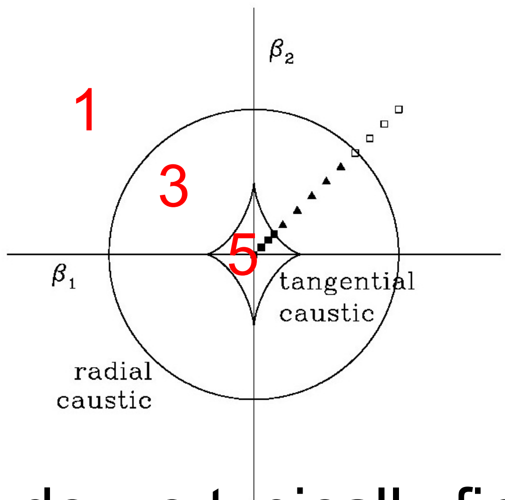
Critical curves and caustics VI

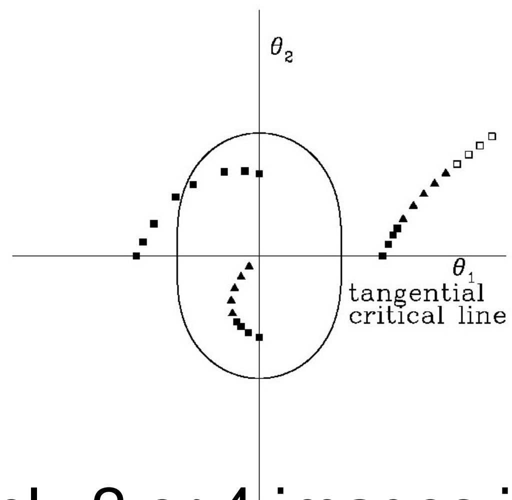


[Courbin et al. 2002]

Critical curves and caustics VII

Caustics separate regions of different image multiplicity



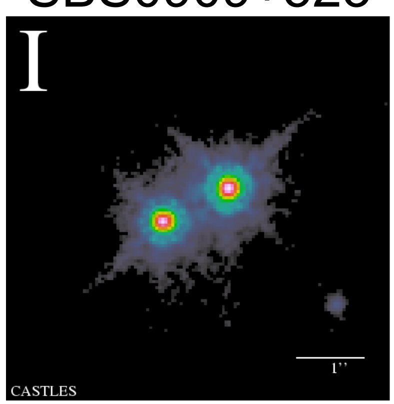


[Image credit:
C. Kochanek]

Why do we typically find only 2 or 4 images in real lens systems? *Ans.: central image is demagnified*

CASTLES

SBS0909+523



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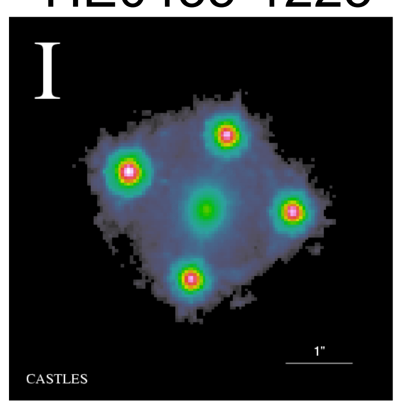
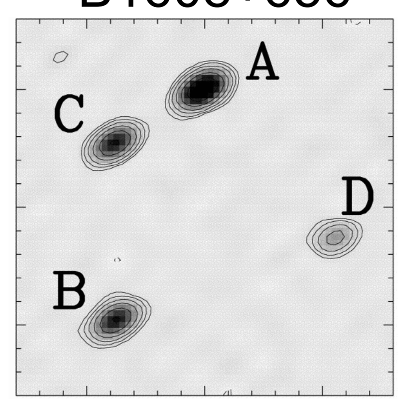


B1608+656

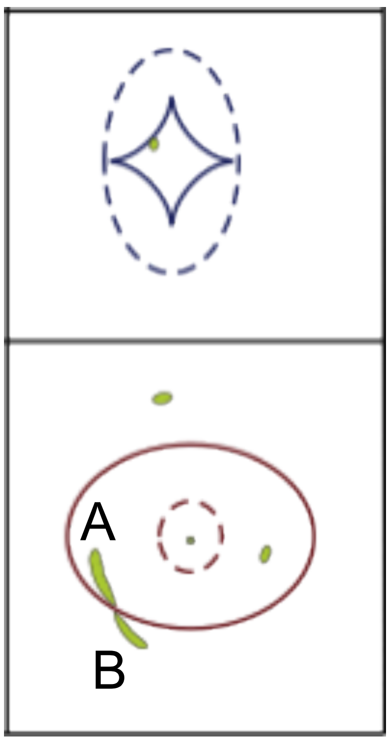


[Fassnacht et al. 2002]

Cusp and fold relations

For smooth mass distribution:

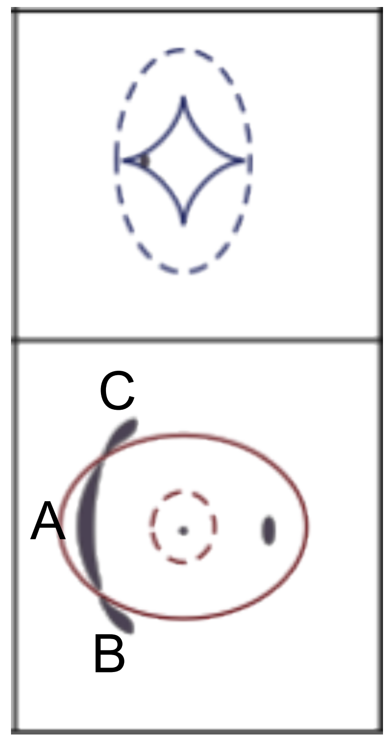
<u>Fold relation</u>



$$\mu_A + \mu_B = 0$$

$$|\mu_A| = |\mu_B|$$

<u>Cusp relation</u>



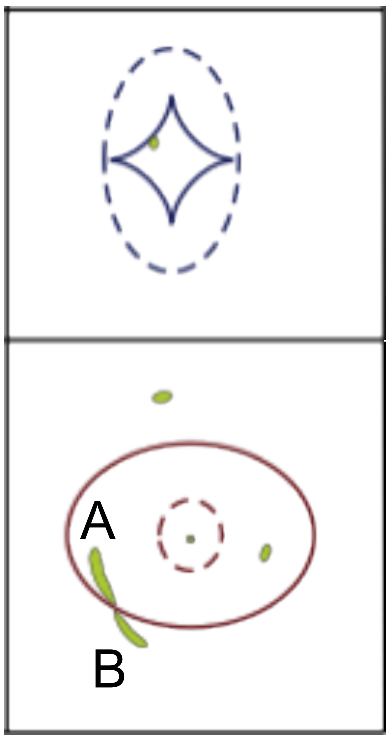
$$\mu_A + \mu_B + \mu_C = 0$$

$$|\mu_A| = |\mu_B| + |\mu_C|$$

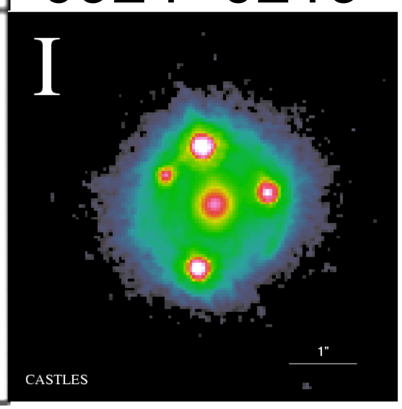
Observed cusp and fold lenses

For smooth mass distribution:

<u>Fold relation</u>



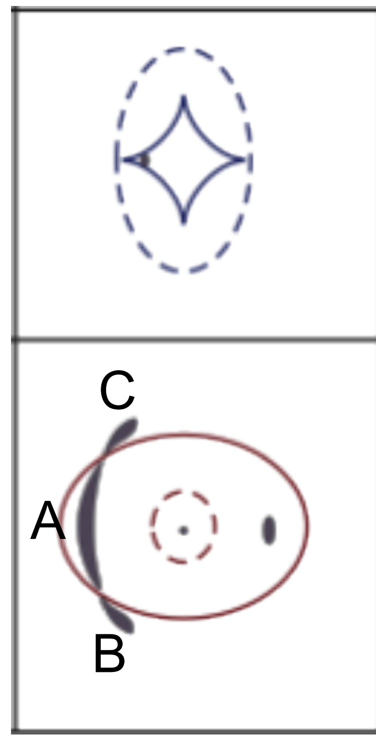
0924+0219



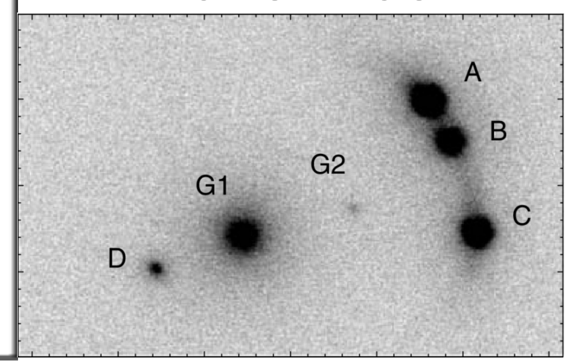
$$\mu_A + \mu_B = 0$$

$$|\mu_A| = |\mu_B|$$

<u>Cusp relation</u>



2045+265

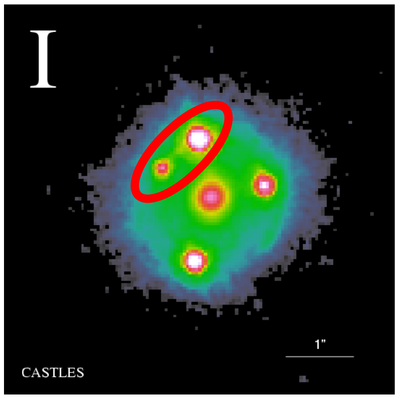


$$\mu_A + \mu_B + \mu_C = 0$$

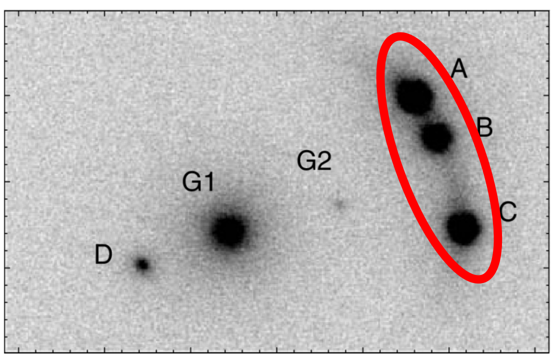
$$|\mu_A| = |\mu_B| + |\mu_C|$$

Violations of cusp and fold relations

0924+0219



2045+265



What causes the violations?

- time delays (variability)
- dust extinction
- microlensing
- substructure

By eliminating the first three causes, lensing provides a unique way to detect dark matter substructure