Cross-Spectral Analysis of Global Oscillation Time Series

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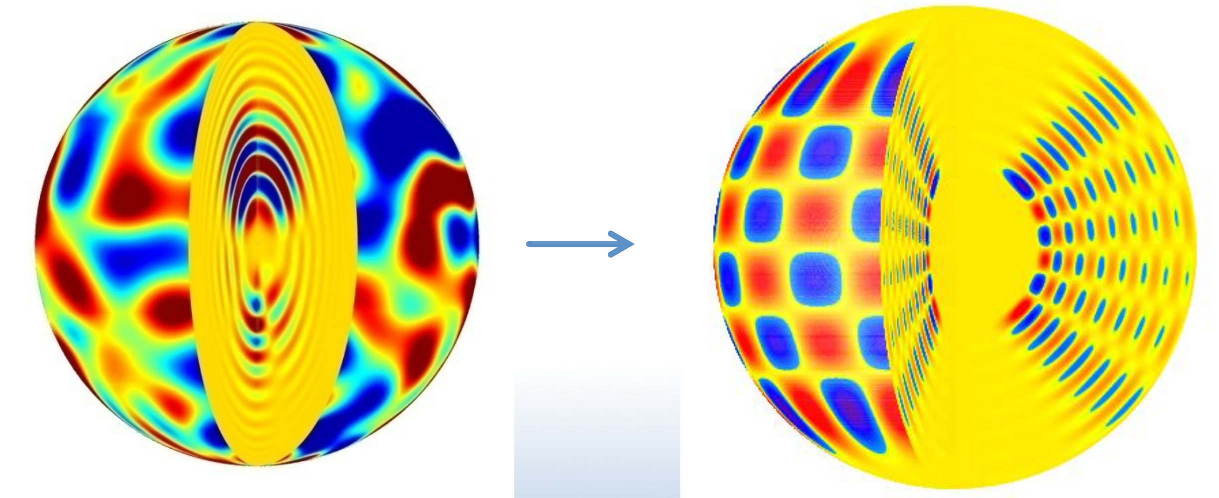
Global Helioseismology

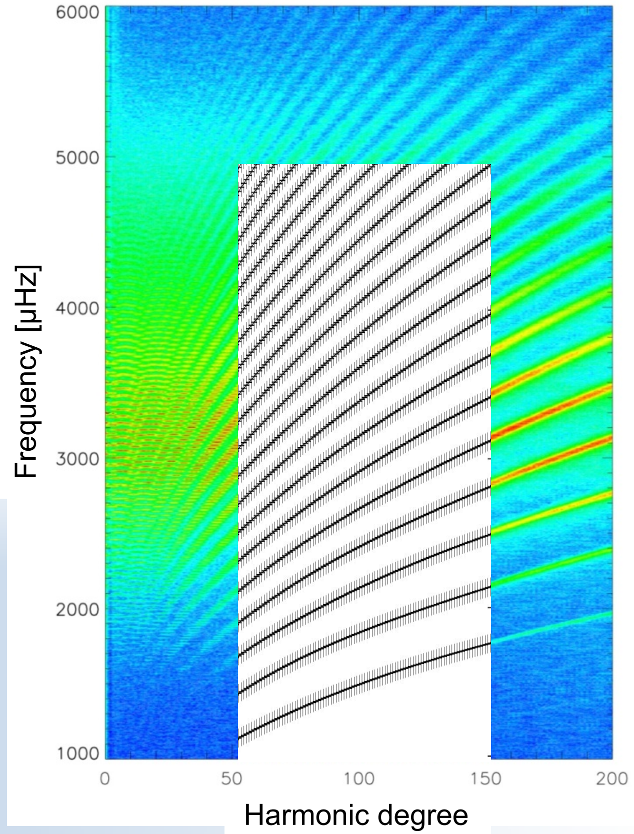
Cross-Spectral Analysis of Global Oscillation Time Series

Many discoveries by using the following concept:

- Decomposing normal standing modes on the basis of spherical harmonics

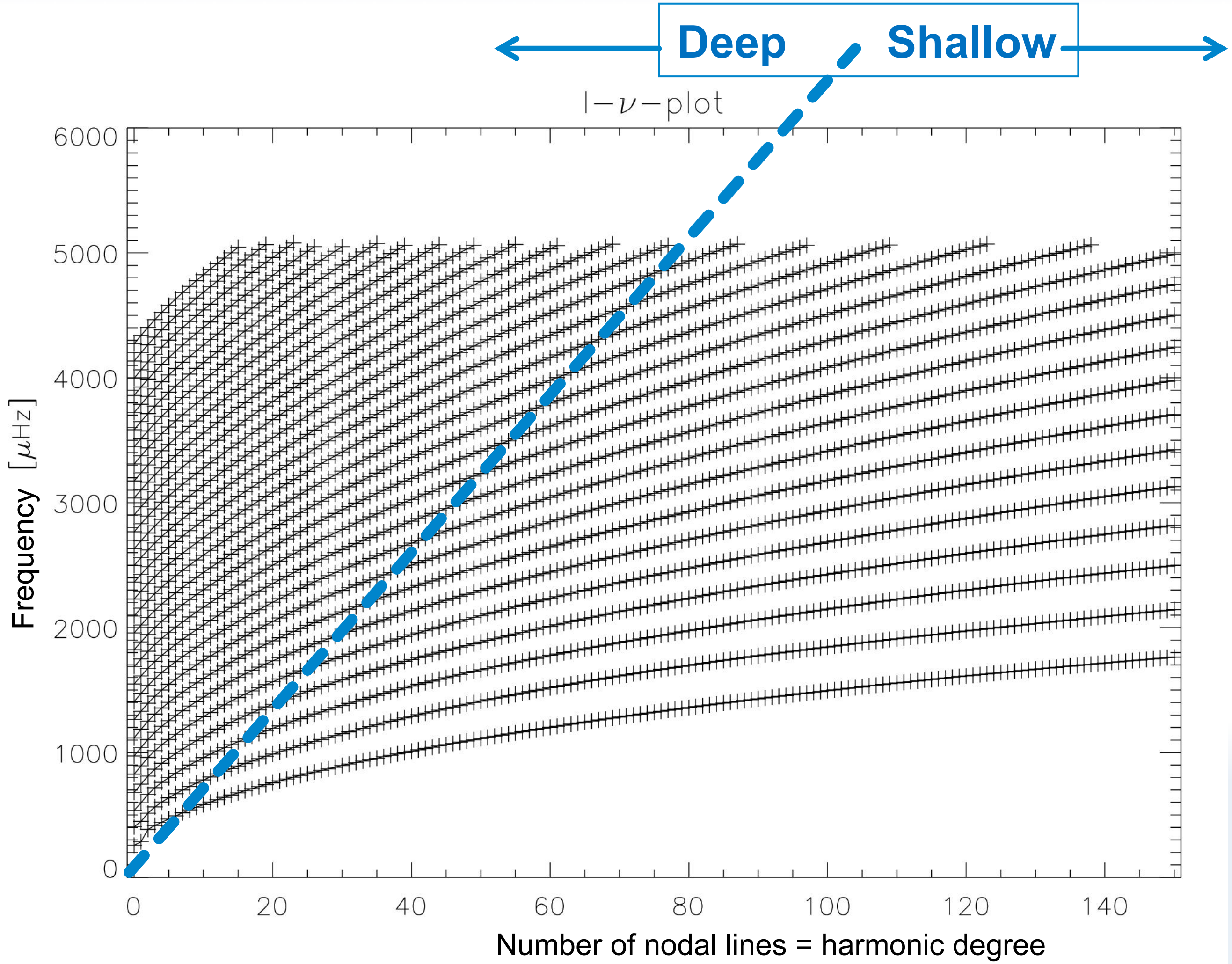
$$\xi(R_\odot,\theta,\phi,t) = Re\left[\sum_{n,l,m} A_{nlm}(t)\xi_{nlm}(R_\odot)Y_{lm}(\theta,\phi)\exp(\mathrm{i}\omega_{nlm}t+\varphi_{nlm})\right]$$





- Results are averages over the entire Sun
  - No longitudinal information
  - Symmetric in latitude across the equator

Penetration Depth of Modes



Quasi-Degenerate Perturbation Theory

Perturbing the equilibrium model with a slow flow (small perturbation)

$$\begin{aligned} -\omega_k^2 \rho_0 \xi_k &= -\nabla p_1 + \rho_0 \mathbf{g}_1 + \rho_1 \mathbf{g}_0 \\ -\omega_k^2 \rho_0 \xi_k - 2\mathrm{i}\omega_k \rho_0 (\mathbf{v}\cdot\nabla)\xi_k &= -\nabla p_1 + \rho_0 \mathbf{g}_1 + \rho_1 \mathbf{g}_0 \end{aligned}$$

$$-\rho_0 \omega_k^2 \xi_k = H_0(\xi_k) + \varepsilon H_1 \xi_k$$

Mode Coupling: Perturbation matrix elements for calculation of new eigenvalues

$$H_{k'k} = \langle \xi_k | 2\mathrm{i}\omega_k \rho_0 (\mathbf{v}\cdot\nabla) | \xi_{k'} \rangle$$

Eigenvalues of perturbation matrix are frequency corrections: $\tilde{\omega}_k = \omega_k + \delta\omega_{1,k}$

Flow Modelling

Decomposition into a
- toroidal flow (includes differential rotation) and
- poloidal flow (includes meridional flow and giant cells)

$$\mathbf{v}(\mathbf{r}) = \sum_{s=0}^{\infty} \sum_{t=-s}^{s} \mathbf{T}_s^t(r;\theta;\phi) + \mathbf{P}_s^t(r;\theta;\phi)$$

where components are expanded in terms of spherical harmonics

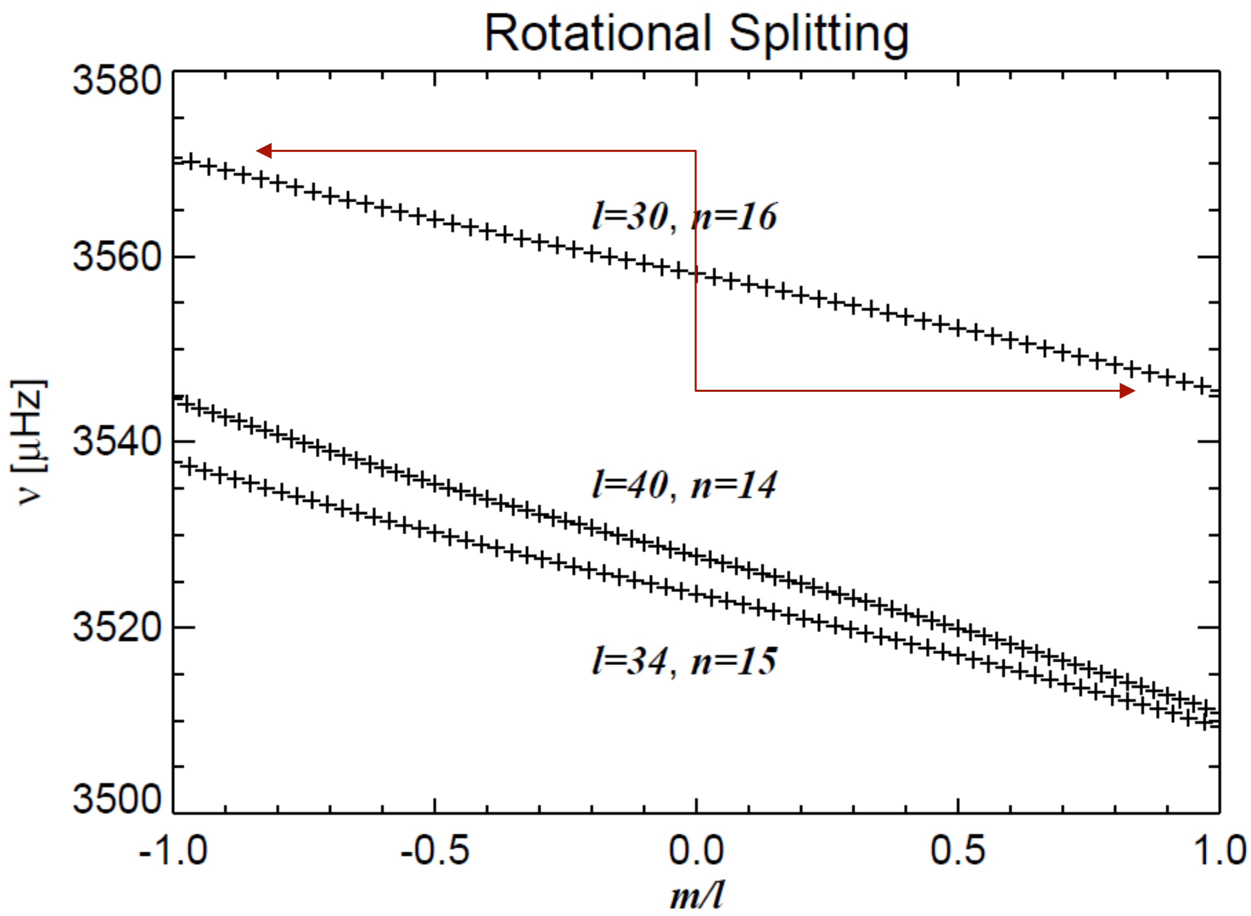
$$\mathbf{T}_s^t(r;\theta;\phi) = -w_s^t(r)\,\mathbf{e}_r \times \nabla_h Y_s^t(\theta;\phi)$$

$$\mathbf{P}_s^t(r;\theta;\phi) = u_s^t(r)\,Y_s^t(\theta;\phi)\,\mathbf{e}_r + v_s^t(r)\,\nabla_h Y_s^t(\theta;\phi)$$

Differential Rotation: Frequency Splitting

Lifting of degeneracies

! “Self-coupling” within multiplets



Differential Rotation: Frequency Splitting

Toroidal Flow: $$\mathbf{v}_0(\mathbf{r}) = \sum_s \sum_{t=-s}^{s} -w_s^t(r)\mathbf{e}_r \times \nabla_h Y_s^t(\theta,\phi)$$

Differential rotation (s odd, t=0) : $\omega_k(m)=\omega_k(m=0)+\delta\omega(m)$

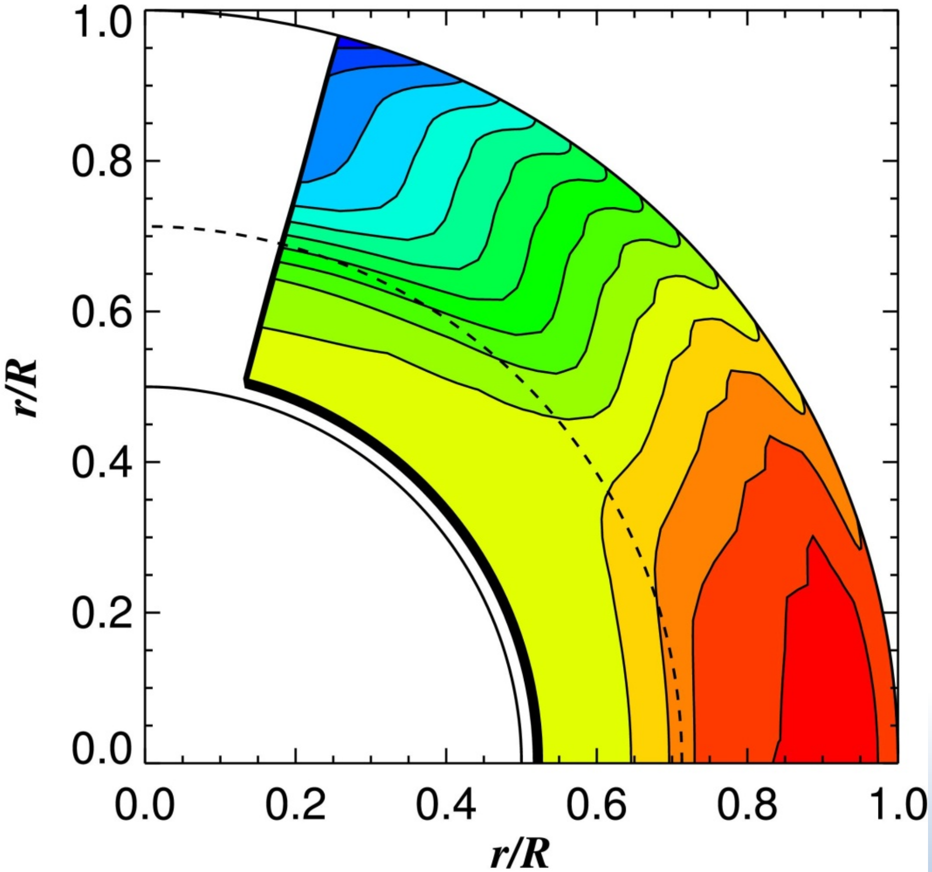
with $\delta\omega(m)=\sum_{s=1,3,5,\ldots} c_{nl,s}\gamma_{nl,s}(m)$

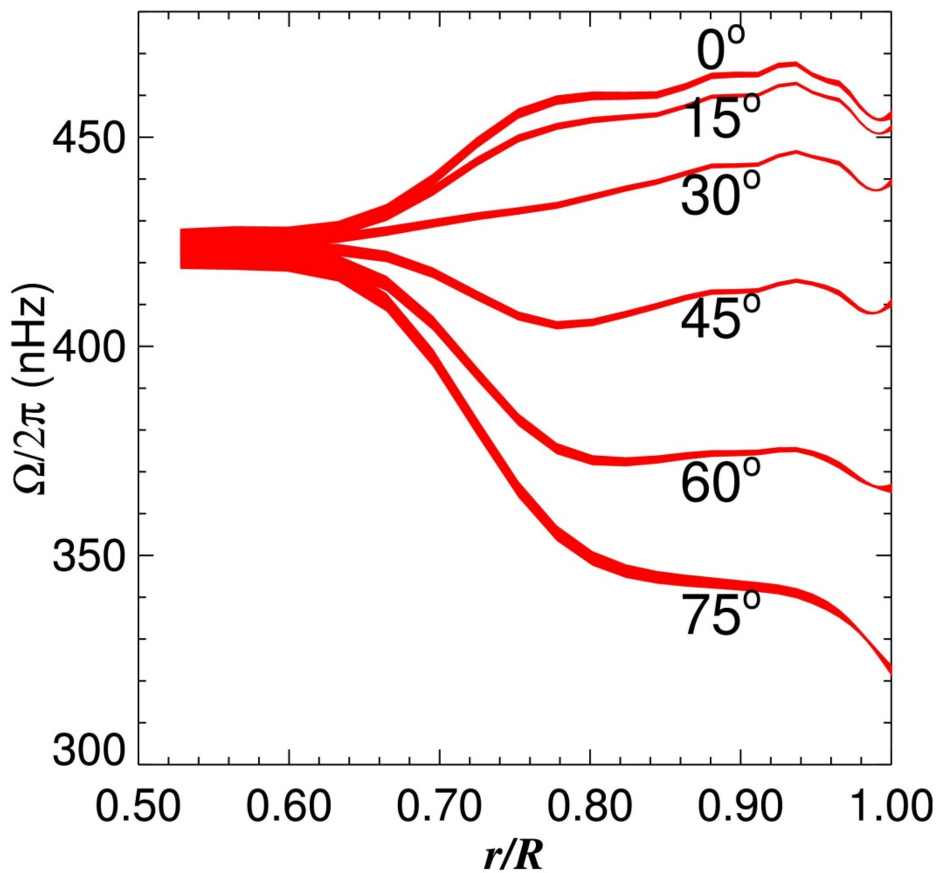
where $c_{nl,s}=s_0^R\ w_s(r)\ K_{nl,s}(r)\ r^2\ dr$

and $\gamma_{nl,s}$ orthogonal functions (Clebsch-Gordon coefficients)

! **Inversion problem for w_s(r)**

Inversion of Frequency Splittings





Poloidal Flows: Additional Frequency Shifts

Poloidal Flows:

$$\mathbf{v}_0(\mathbf{r}) = \sum_{s=0} \sum_{t=-s}^{s} u_s^t(r) Y_s^t(\theta,\phi)\mathbf{e}_r + v_s^t(r)\nabla_h Y_s^t(\theta,\phi)$$

$s \neq 0$, $t \neq 0$: giant cells

$s \neq 0$, $t = 0$: meridional flow

Frequency shifts:

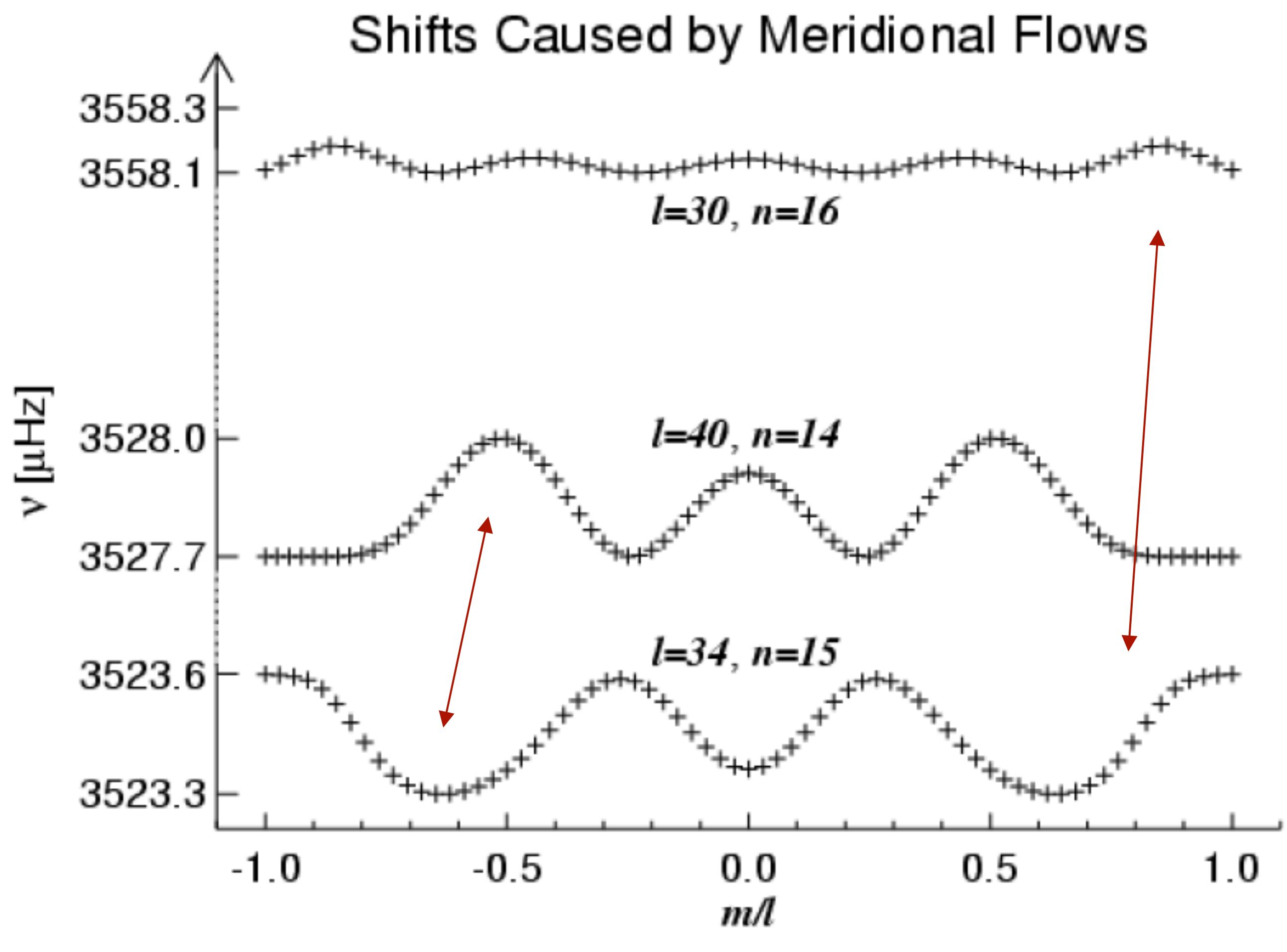
$\delta\omega(m)=\sum_{s+l+l' \text{ even}} (c_{nn'll',s}\, \gamma_{nn',ll',s}(m))^2$

basis functions are squared Clebsch-Gordon coefficients ! Orthogonality?

No global helioseismology inversions with the existing frequency analysis tools

Frequency Shifts caused by Meridional Flow

- V_{max}=100 m/s
- s=8, t=0



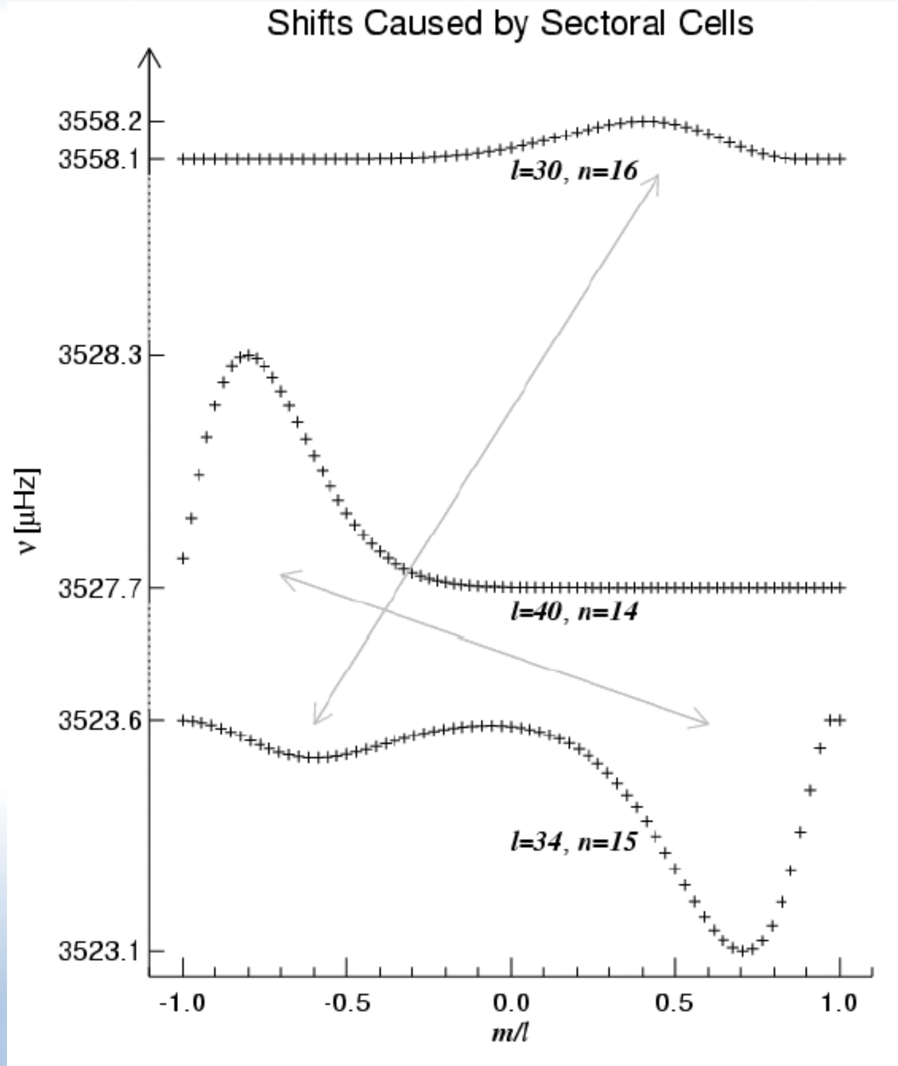
Frequency Shifts from Giant Cells

- V_{max}=100 m/s
- s=8, t=8

Additional frequency shifts

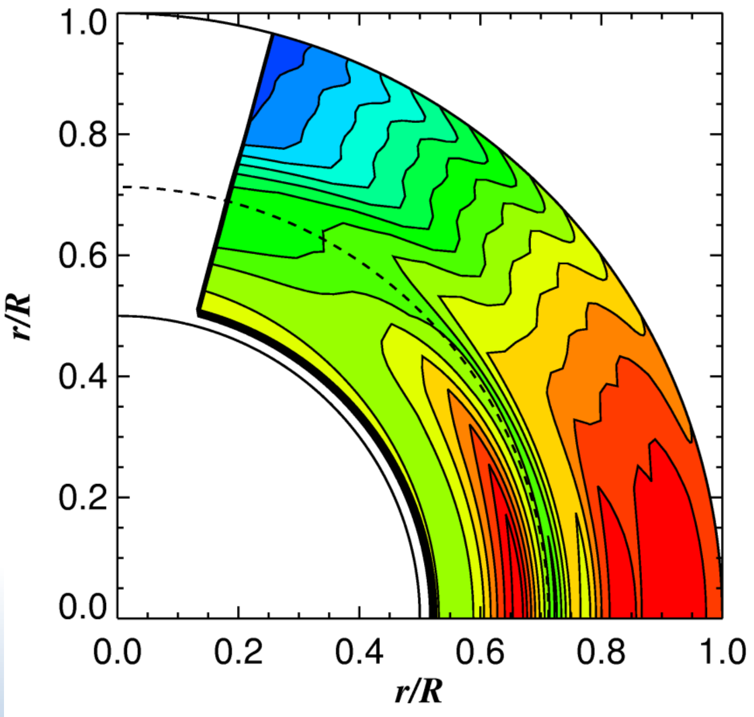
Giant cells & meridional circulation leave signature in global data

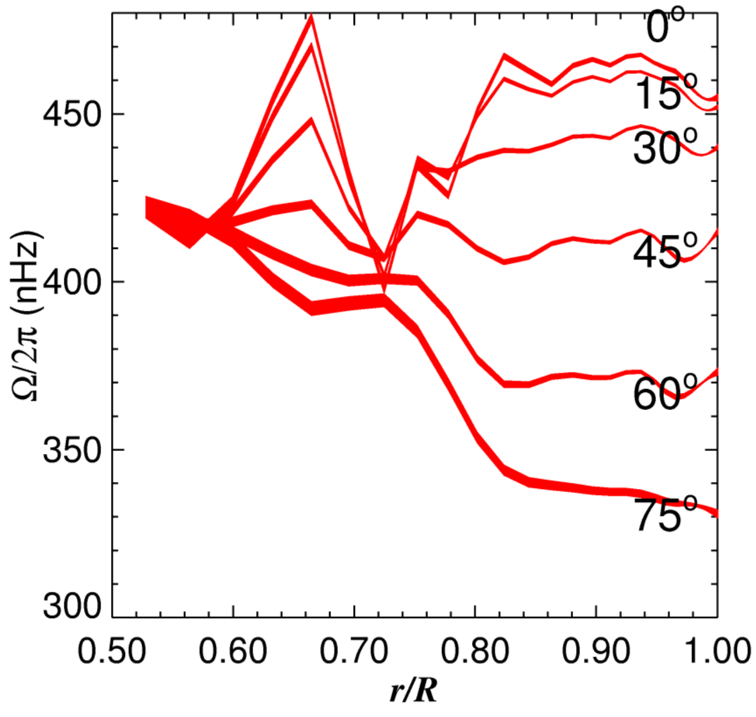
Real effect is very small in comparison to rotational splitting.



Effect of Giant Cells on Frequency Inversions for Rotation

V= 10m/s 60m/s



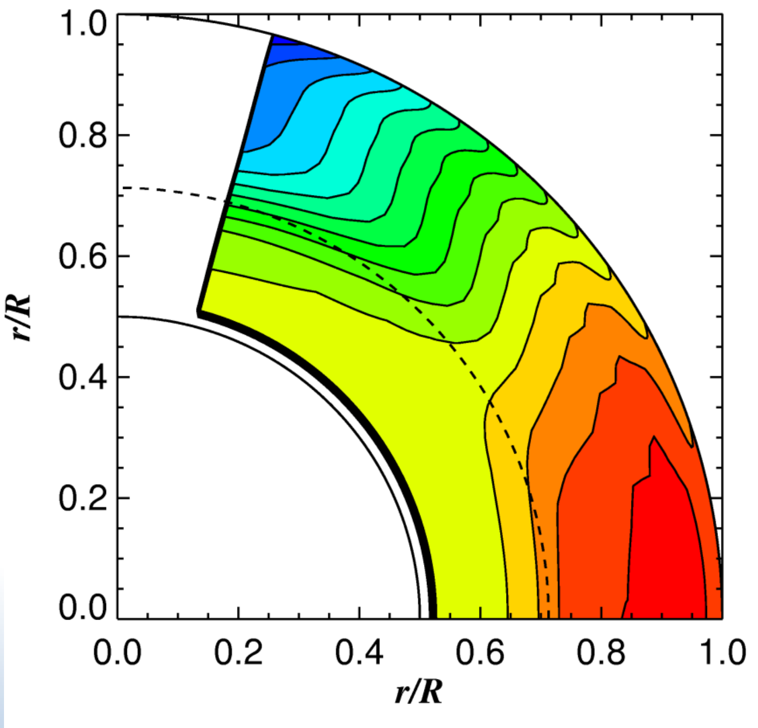


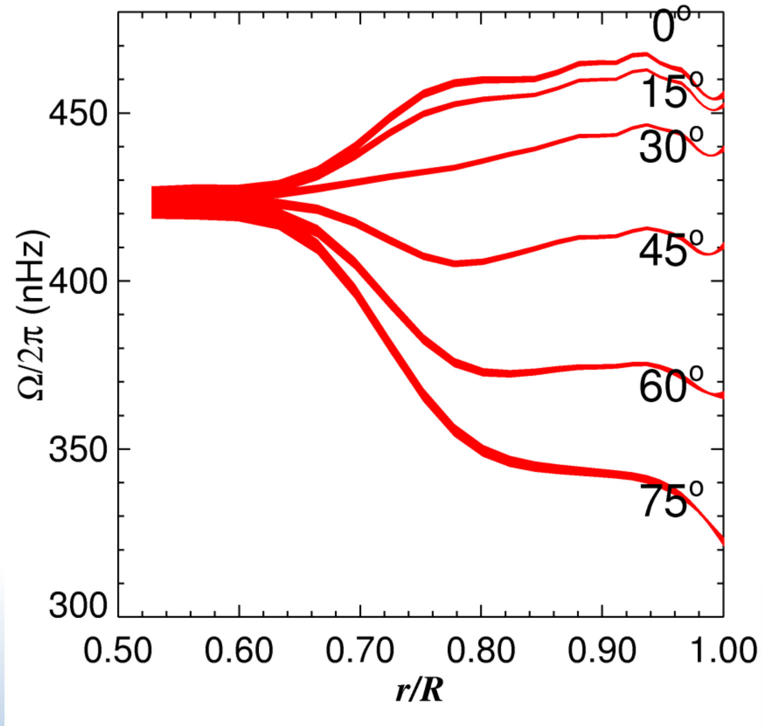
Given limits for giant cell velocities in the Sun do not cause a problem.
What about stars?

(Roth et al., 2002, A&A)

Effect of Meridional Flow on Frequency Inversions for Rotation

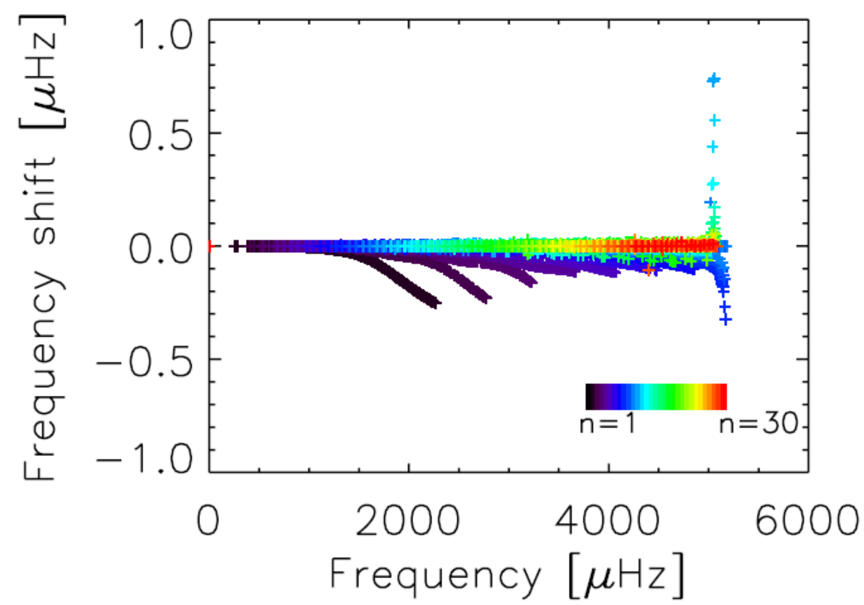
V= 100m/s





(Roth et al., 2002, A&A)

Overall Effect of Poloidal Flows



The effect of large-scale poloidal flows on solar oscillations

Perturbation theory:

in first order no effect on the frequencies, only in higher orders

Resulting average effect:

Frequencies corrected down

Stix & Zhugzhda 1994:

in the order to explain discrepancies between theoretical and observed frequencies

→giant cells & meridional flow not measurable with global helioseismology?

But: **first order perturbations in the eigenfunctions**

Observed oscillations are no pure eigenstates any more

Global Helioseismology – Perturbation Theory

Perturbation theory applied to the oscillation equations of p modes:

> coupling of modes in a “neighborhood” K_k of a target mode k=(n,l,m)

$$\boldsymbol{v}_k(r,\theta,\phi,t) = \underbrace{\alpha_k(t)}_{\text{osc. amplitude}}\,\underbrace{\boldsymbol{\xi}_k(r,\theta,\phi)}_{\text{perturbed eigenfunction}} = \alpha_k(t)\sum_{k'\in K_k} c_{kk'}\underbrace{\boldsymbol{\xi}^0_{k'}(r,\theta,\phi)}_{\text{unperturbed eigenfunction}}\,,$$

(Lavely & Ritzwoller 1992)

> $c_{kk''}$ – coupling coefficient, 1. order approximation

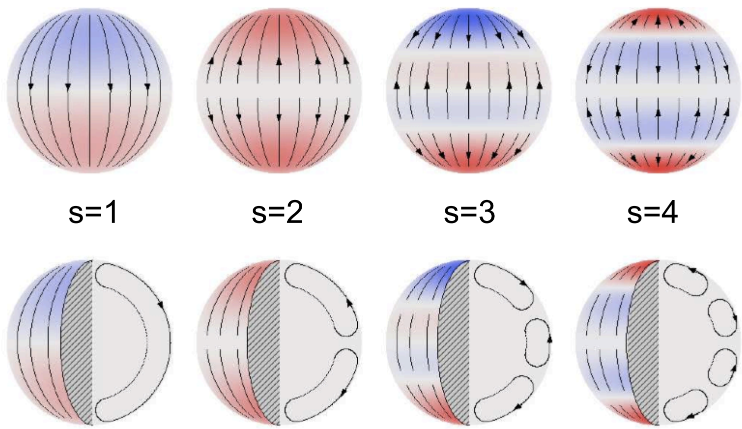
$$c_{kk'} \approx \begin{cases} 1 & \text{for} \quad k' = k \\ \frac{H_{k'k}}{\omega_k^2-\omega_{k'}^2} & \text{for} \quad k' \in K\backslash\{k\} \end{cases}$$

> $H_{k'k}$ – matrix element of mode coupling between mode k,k‘ (≈ coupling strength):

$$H_{k'k} = -2\,\mathrm{i}\,\omega_{ref}\int \rho_0\,\overline{\boldsymbol{\xi}^0_{k'}}\cdot\underbrace{(\boldsymbol{u}\cdot\nabla\boldsymbol{\xi}^0_k)}_{\text{advection of acoustic wave}}\,d^3\boldsymbol{r}$$

Schad, Timmer, Roth, 2011, Astrophys. J. 734, 97

Global Helioseismology – Perturbation Theory



(Figure source: D. Hathaway, NASA)

$$\boldsymbol{u}(\boldsymbol{r}) = \sum_{s=1}^{\infty} \left[\underbrace{u_s^0(r) Y_s^0(\theta,\phi) \boldsymbol{e}_r}_{\text{radial component}} + \underbrace{v_s^0(r) \partial_\theta Y_s^0(\theta,\phi) \boldsymbol{e}_\theta}_{\text{horizontal component}} \right]$$

Mass conservation: $\boxed{\rho_0 r s(s+1) v_s^0 = \partial_r (r^2 \rho_0 u_s^0)}$

→ coupling matrix element $\quad H_{k'k} = H_{n'l',nl}(m) = \mathrm{i}\,\omega_{ref} \sum_s b^s_{n'l',nl} \mathcal{P}^s_{l'l}(m)$,

complete set of orthogonal polynomials: $\quad \mathcal{P}^s_{l'l}(m) := (-1)^{-m} \begin{pmatrix} l' & s & l \\ -m & 0 & m \end{pmatrix}$,

b-coefficients: $\boxed{b^s_{n'l',nl} = \int_0^R \rho_0(r) K_s^{n'l',nl}(r) u_s^0(r) r^2 dr}$

→ knowing $b^s_{n'l'nl}$ one can infer the radial flow strength $u_s(r)$!

Schad, Timmer, Roth, 2011, Astrophys. J. 734, 97

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Observable Effect on Global Oscillations

SHT of Dopplergrams:

$$o_{l'm'}(t) = \int \overline{Y_{l'}^{m'}}(\theta,\phi) W(\theta,\phi) v_D(\theta,\phi,t) d\Omega\,,$$

FT – frequency domain ⇓ $\tilde{o}_{l'm'}(\omega)$

$$= \sum_k \alpha_k(t) \sum_{k'' \in K_k} c_{kk''} L_{k'k''}\,,$$

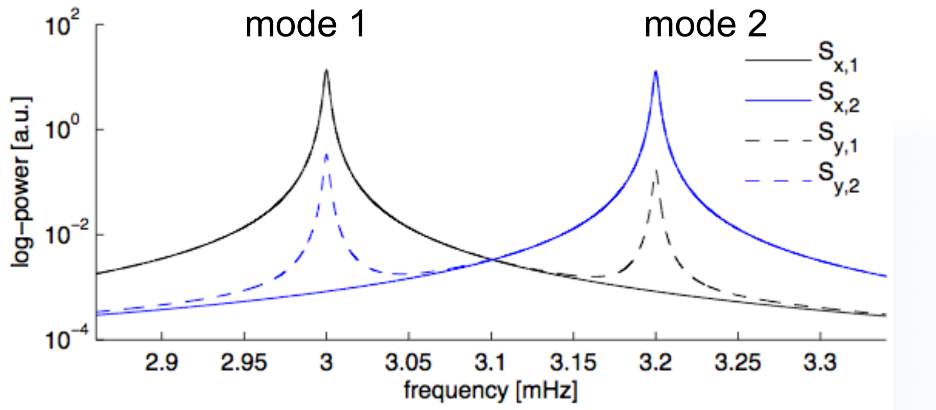
$L_{k'k''}$ - leakage matrix

Observable: amplitude ratio between target mode k=(n,l,m) and coupling modes k' in K_k:

$$y_{lm\,l'm}(\omega_{nlm}) := \frac{\tilde{o}_{l'm}(\omega_{nlm})}{\tilde{o}_{lm}(\omega_{nlm})} \approx \frac{\sum_{k'' \in K_k} c_{kk''} L_{k'k''}}{\sum_{k'' \in K_k} c_{kk''} L_{kk''}} \in \mathbb{C}$$

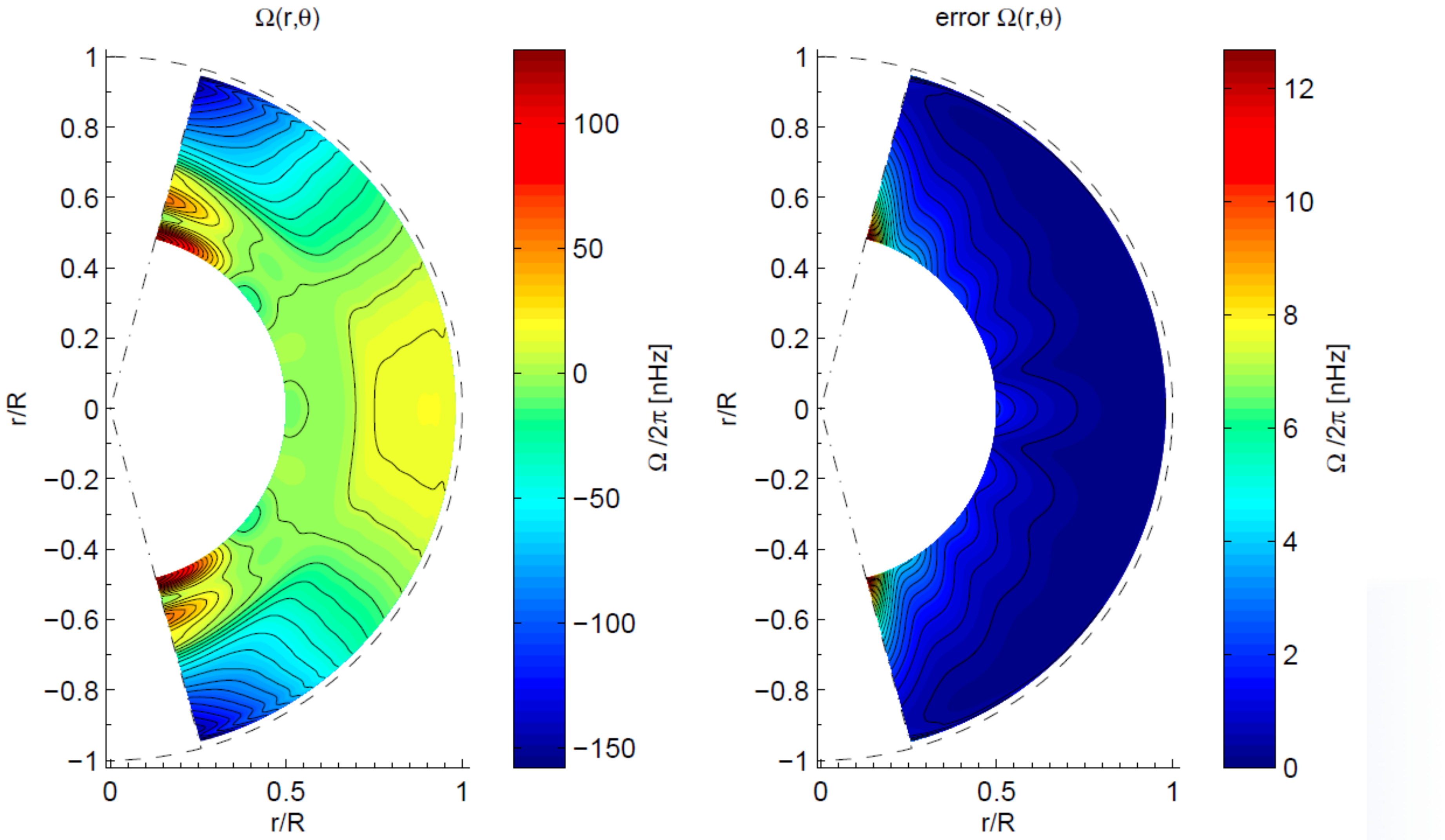
target mode (↑ denominator $\tilde{o}_{lm}(\omega_{nlm})$)

„ ratio between global oscillations in the frequency domain that corresponds to coupling modes"



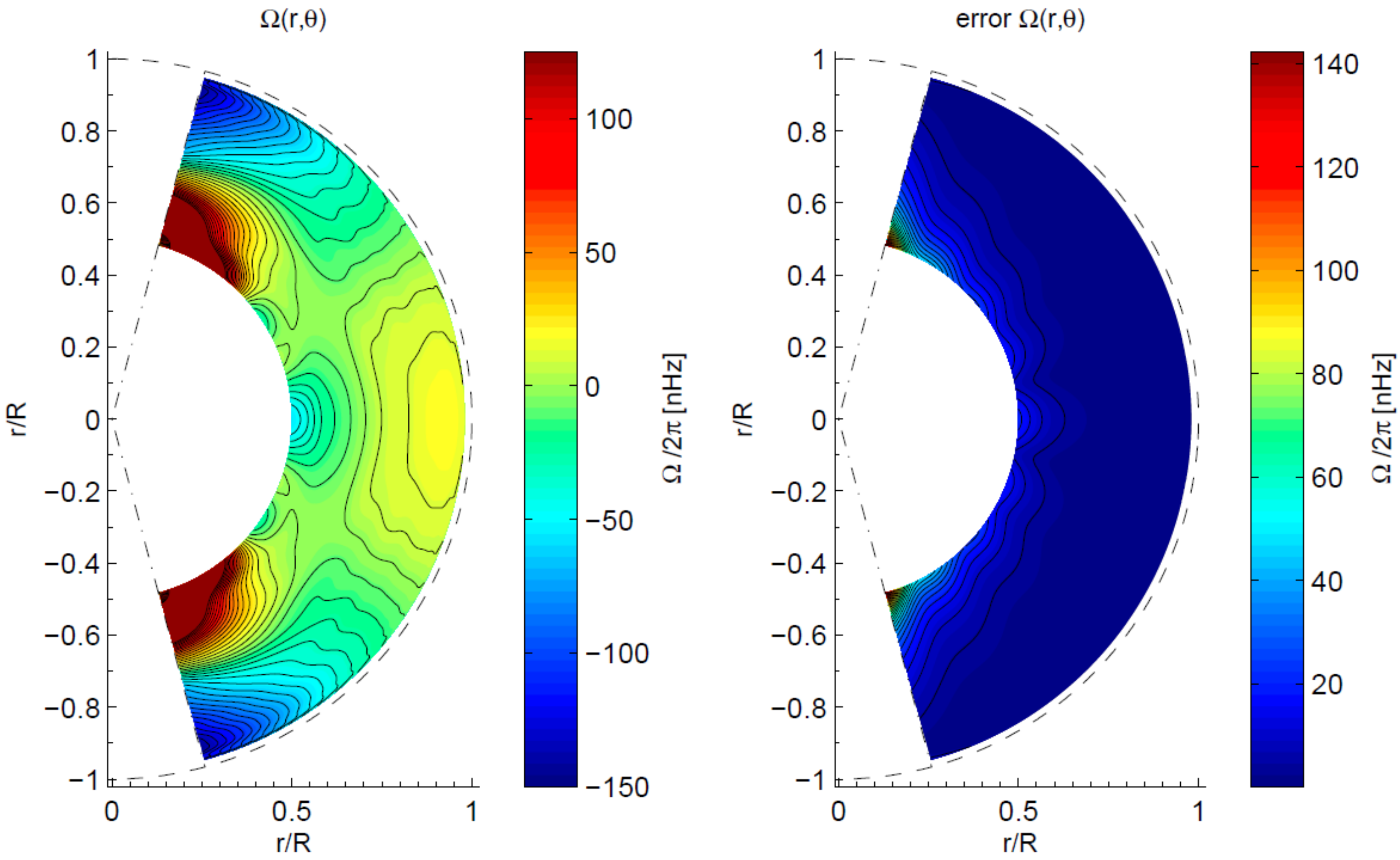
Schad, Timmer, Roth, 2011, Astrophys. J. 734, 97

Inversion for Rotation from Frequency Splittings



Inversion for Rotation from Amplitude Ratios

Cross-Spectral Analysis of Global Oscillation Time Series



Global Inversion Method for the Meridional Flow

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Operation scheme:

- estimation of amplitude ratios $y_{kk'}$ for the multiplets > cross-spectral analysis / gain
- estimation of b-coefficients $b^s_{kk'}$ > least squares fitting routine
- inversion of radial flow strength $u_s(r)$ > e.g. SOLA inversion method (as used for diff. rotation)
- reconstruction of the horizontal flow strength $v_s(r)$ from $u_s(r)$ > e.g. polynomial fit

$$\boxed{\rho_0 r s(s+1) v_s^0 = \partial_r (r^2 \rho_0 u_s^0)}$$

Application to MDI data

Cross-Spectral Analysis of Global Oscillation Time Series

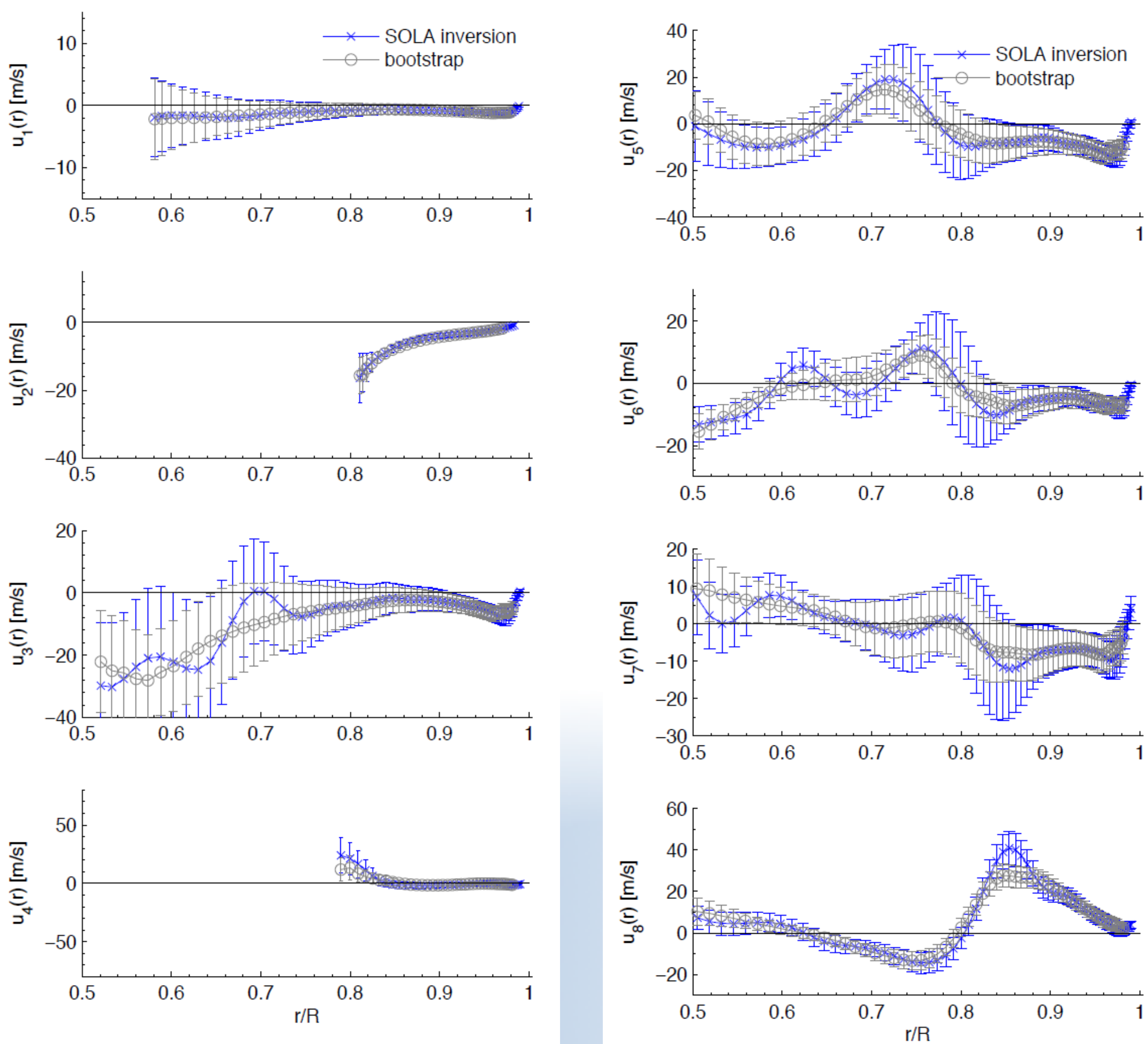
MDI data: 2004 – 2010
Harmonic degree:
1 ≤ l ≤ 200

Radial flow for different components

Deepest depth depends on degree s
(between 0.5 and 0.8 R)

s=2 and s=8
to be studied in detail

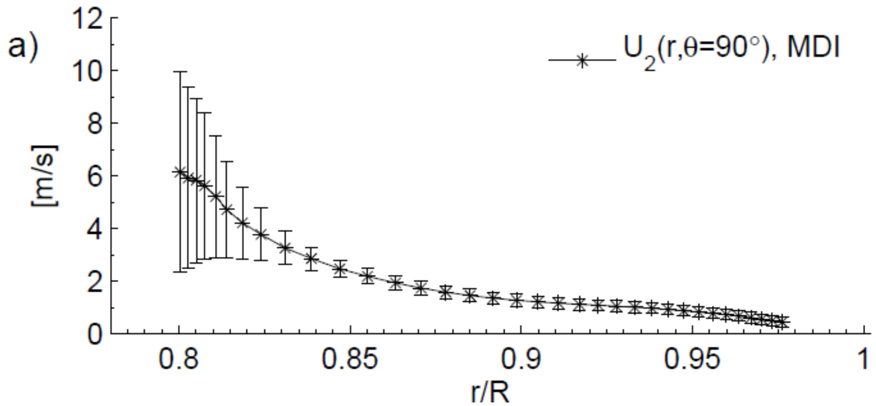
(Schad et al., 2013, ApJL)



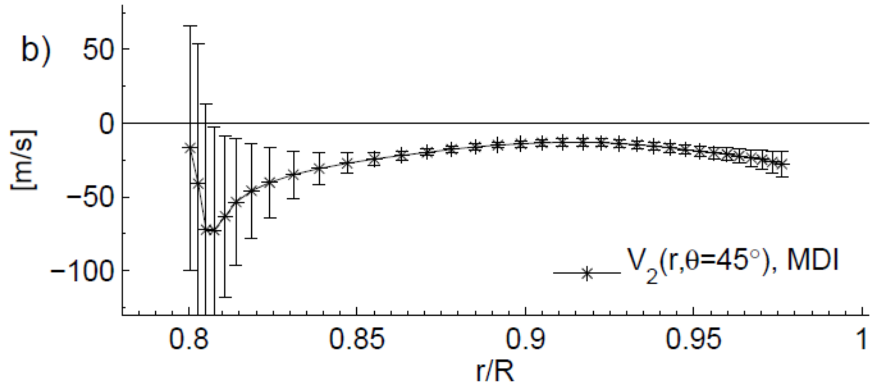
Application to MDI Data – Results for s=2

Cross-Spectral Analysis of Global Oscillation Time Series

Radial flow strength U_2



Horizontal flow strength V_2



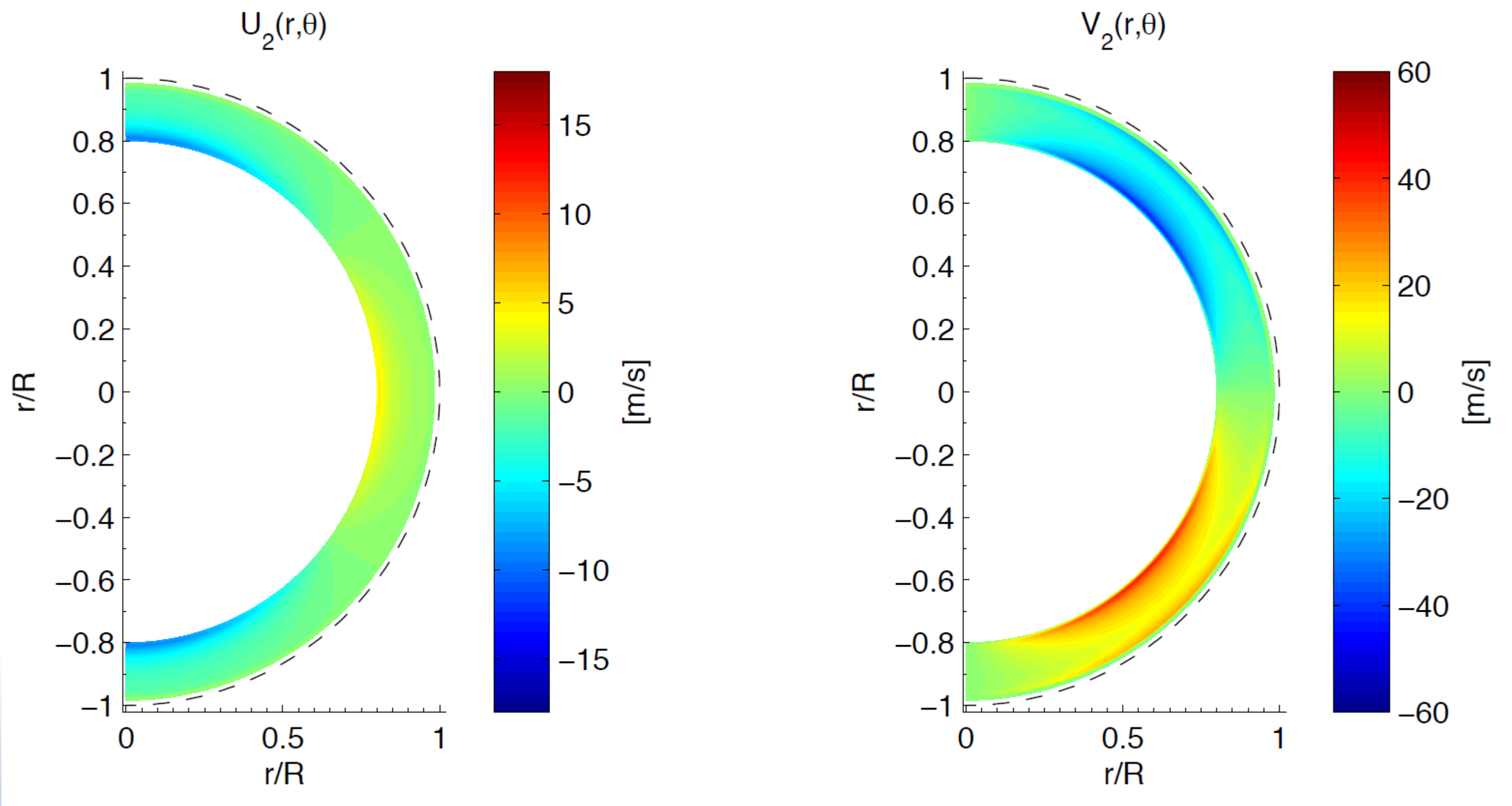
> inversions for $0.82 \leq r/R \leq 0.97$ (depth 20.9 – 125.3 Mm)
> u_2 grows ≈ linear with depth
> v_2 ≈ constant with r
> 1σ-error u_2 : 0.6 – 12 m/s
> 1σ-error v_2 : 2.1 – 100 m/s

> no return flow within $0.82 < r/R < 0.97$!

(Schad et al., 2012, AN)

Application to MDI Data – Results for s=2

Cross-section through radial & horizontal flow profile with latitude (s=2):

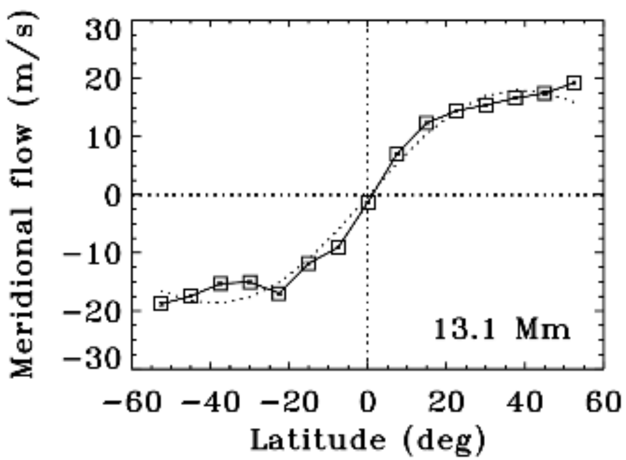


→poleward directed horizontal flow

Comparison with Ring Diagram Analysis (for s=2)

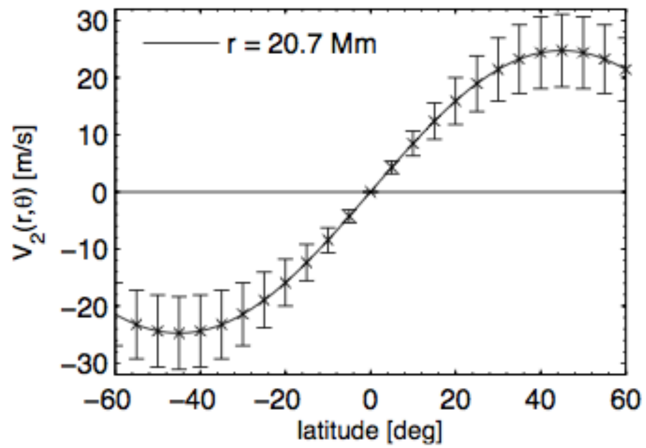
Horizontal flow component of s=2 at 20.7 Mm depth:

Ring-diagram analysis



(Komm et al., 2005, ApJ)

Global inversion method

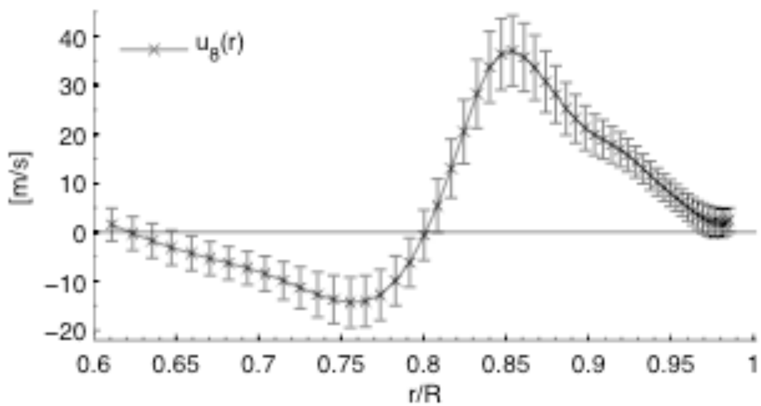


→max. horizontal flow near surface at θ=45°: $V_2 \approx 28 \pm 9$ m/s

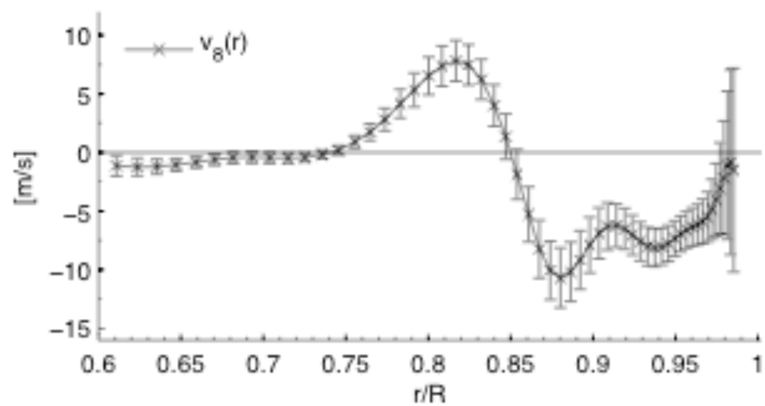
(Schad et al., 2013, ApJL)

Application to MDI Data – Results for s=8

radial flow strength u_8



horizontal flow strength v_8



→ inversions for $0.61 \leq r/R \leq 0.984$ (depth 11 – 271 Mm)
→ two flow cells in depth

(Schad et al., 2013, ApJL)

Comparison with Ring Diagram Analysis (for s=8)

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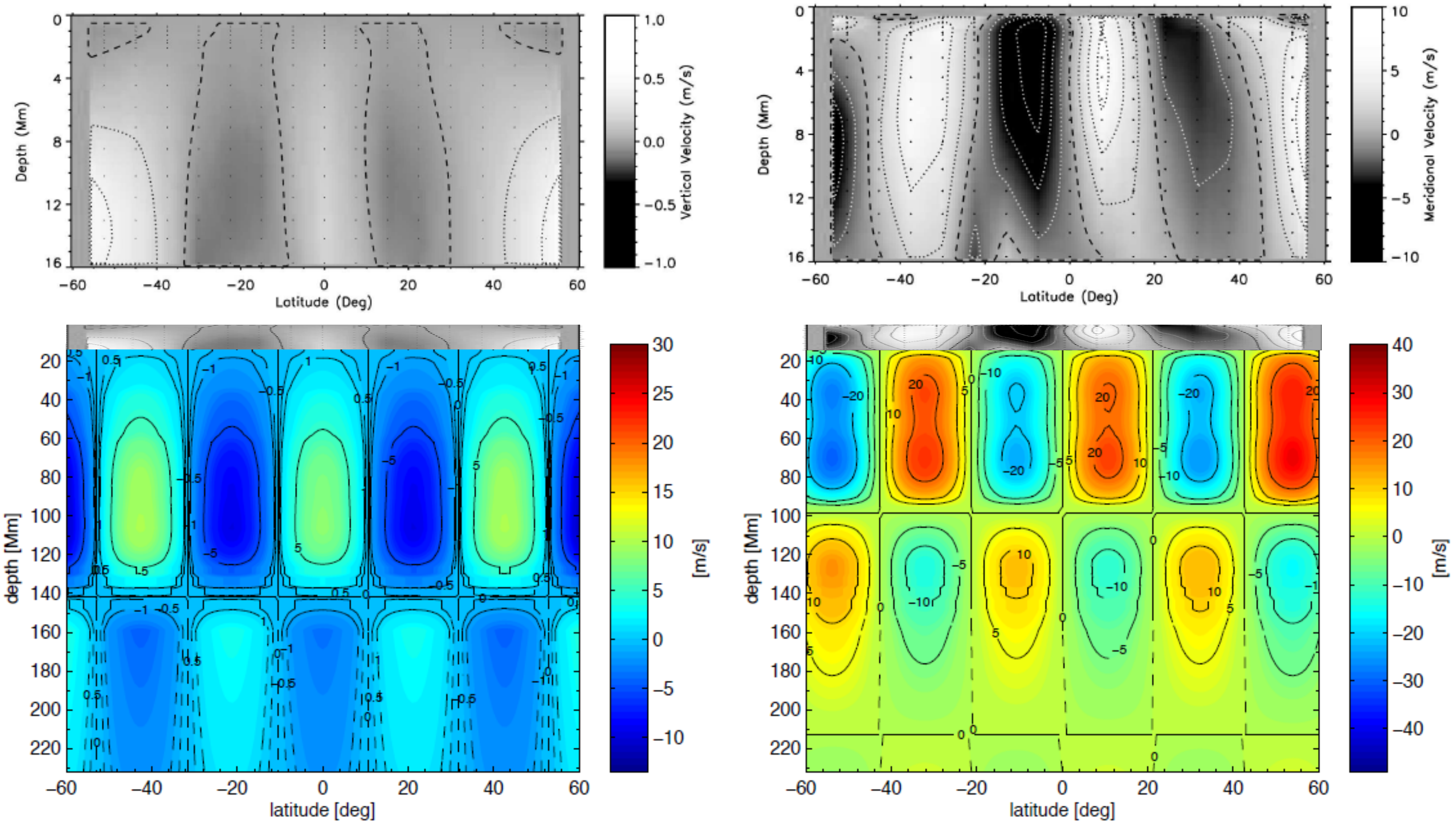
Radial flow | Horizontal flow

Ring-diagram analysis (Komm et al., 2005, ApJ)

Global inversion method (Schad et al., in prep.)

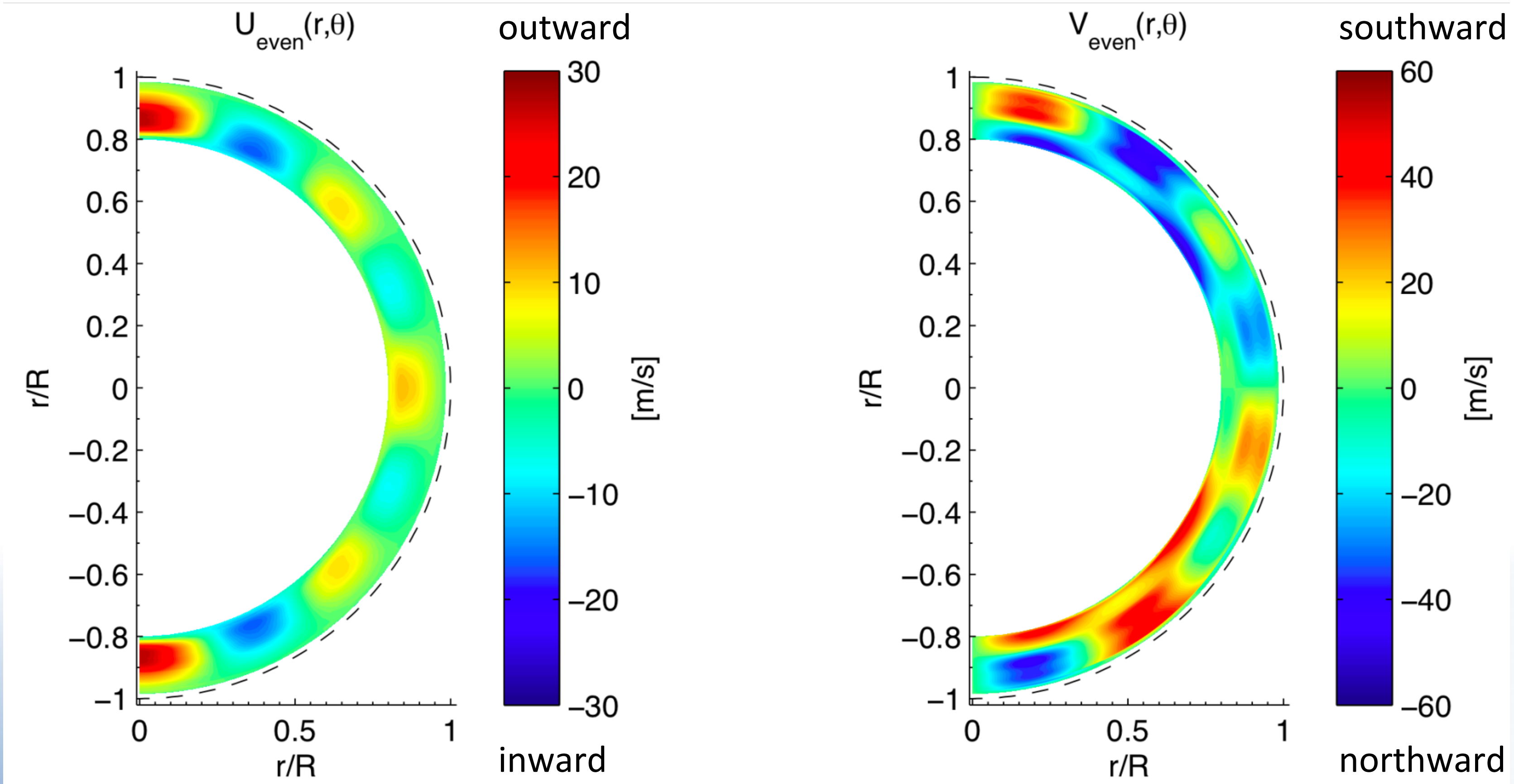


Ring-diagram analysis (0.6–16 Mm depth; 1 year average);
Global method (13–271 Mm depth; 6 years average)

Result of banded flow pattern?

Global Meridional Flow Measurements

Overall Result: Inversion result including all significant flow components in a depth range of 13 – 141 Mm



Clear evidence for multiple cells in depth and latitude

(Schad et al., 2013, ApJL)

Origin for Limitations in Deep Probing

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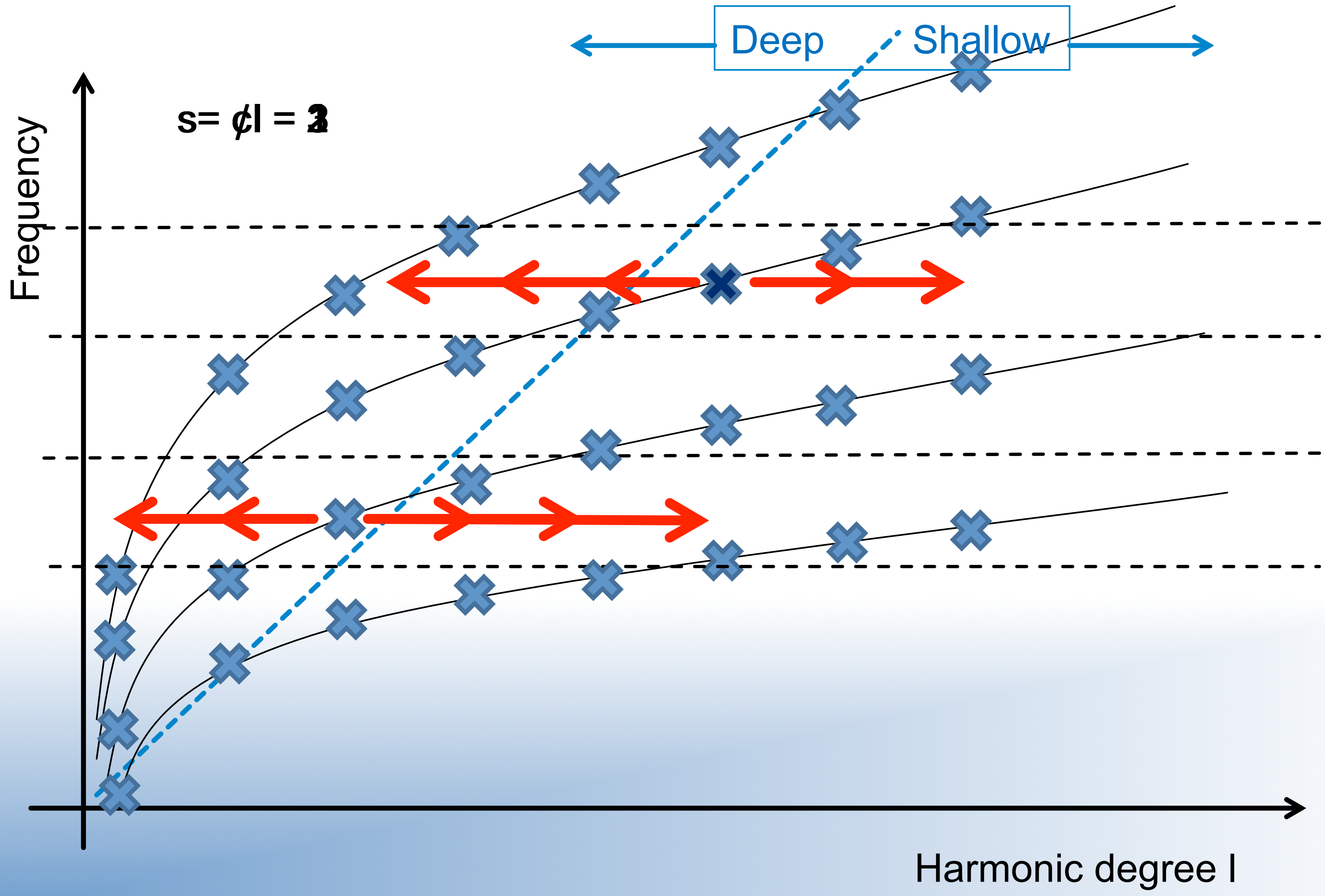
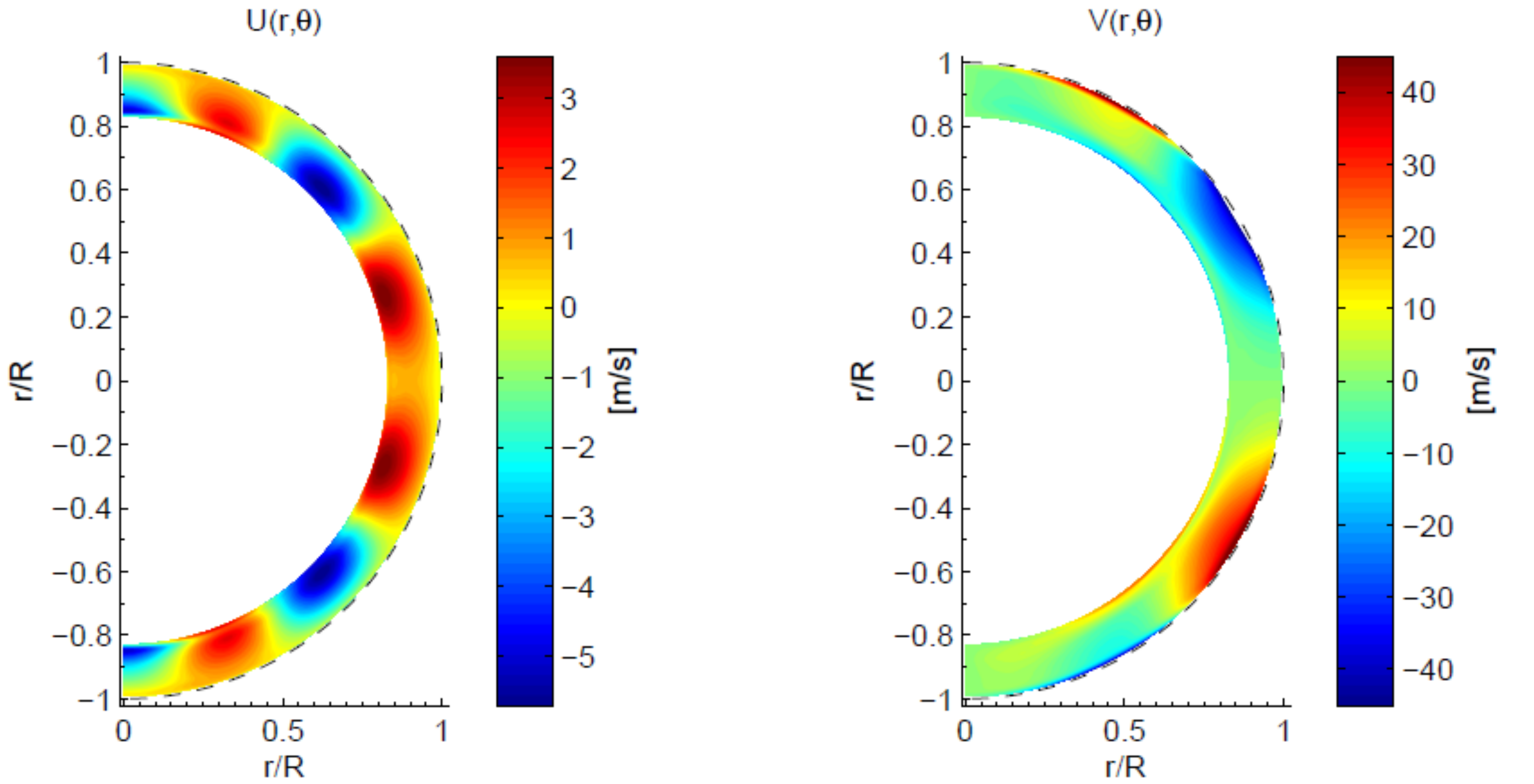


Ignoring Leakage



Knowledge about leakage matrix is crucial.
Ignoring leakage results in plausible but systematically wrong results

Conclusions

- Cross-spectral analysis is a new global helioseismic tool
  - Possible on the Sun (not necessarily on stars) because of long time series needed

- Measurements of flows in the solar interior:
  - Rotation
  - Meridional Flow

- Important Input:
  - Leakage Matrices -> to be part of the peak-bagging tool?

- To Do: Studying systematic effects
  - Leakage Matrices
  - Center-to-limb variation of MDI line (¯$_{nl}$(z); eigenfunctions needed)