High energy physics and inflation as a tool to see it

Lecture 2

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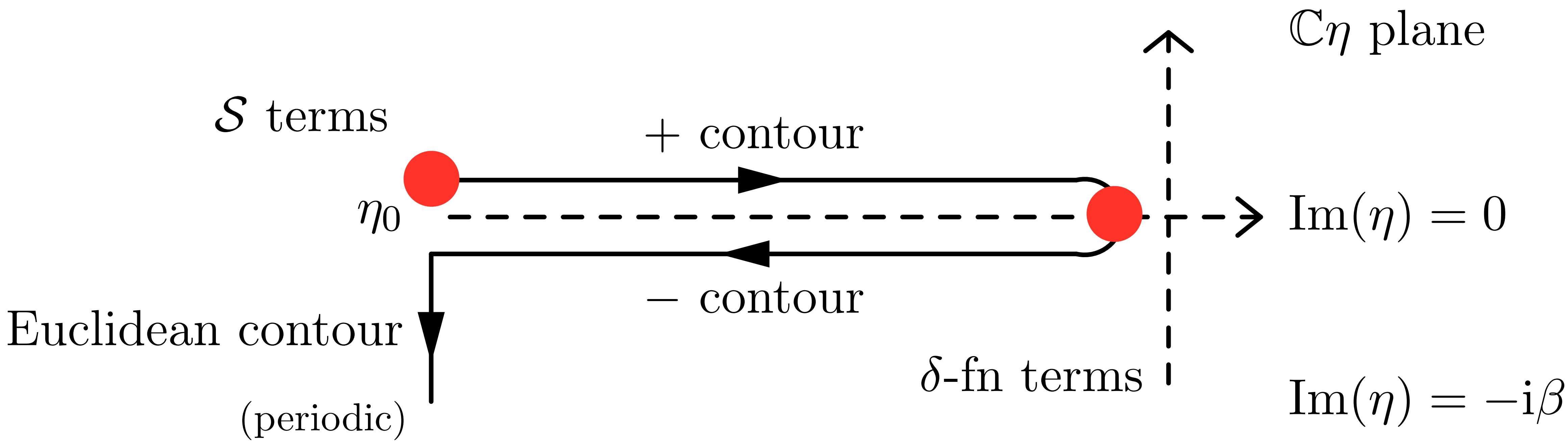
To simplify the notation, it is helpful to consolidate the + and - fields into a single integral over a contour.

We also relabel A → + and B → -



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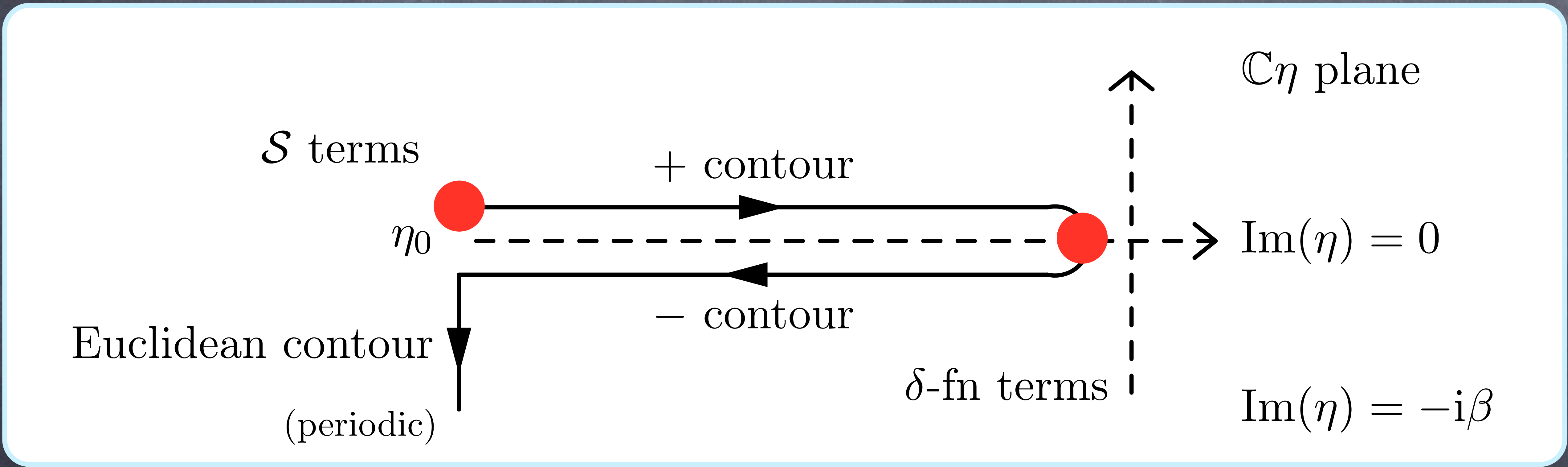

If we send $\eta_0 \to -\infty$, we get Schwinger's theory (vacuum bcs in the infinite past)

If we send $\beta \to \infty$, we get the Gell-Mann / Low theorem. This says we pick out the lowest energy state, ie., the true vacuum

Another representation which is often used is to collect the + and - fields into a matrix. Then it is just like having multiple fields with a weird action

$$\phi = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} \qquad \begin{pmatrix} ++ & +- \\ -+ & -- \end{pmatrix}$$

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$$\boldsymbol{\phi} = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} \qquad \begin{pmatrix} ++ & +- \\ -+ & -- \end{pmatrix}$$

At the quadratic level, we get a matrix derivative operator

$$\exp\left\{ -\frac{\mathrm{i}}{2} \int \mathrm{d}^3x \, \mathrm{d}\eta \; a^4 \, \boldsymbol{\phi} \cdot \begin{pmatrix} \triangle & \\ & -\triangle \end{pmatrix} \cdot \boldsymbol{\phi} + \delta\text{-fn terms} \right\}$$

the 2-point functions are obtained by inverting this operator

$$\mathrm{i}a^4 \begin{pmatrix} \triangle & \\ & -\triangle \end{pmatrix} \begin{pmatrix} G^{++} & G^{+-} \\ G^{-+} & G^{--} \end{pmatrix} = \delta(\eta - \tau)\delta(\boldsymbol{x} - \boldsymbol{y})$$

In the Minkowski vacuum, the boundary conditions at η_0 require that G^{++} is negative frequency (positive energy) and G^{-+} is positive frequency (negative energy)

$$\mathrm{i}\frac{\partial}{\partial t}G^{++}(\boldsymbol{k})|_{\eta_0} = -\omega_{\boldsymbol{k}}G^{++}(\boldsymbol{k})|_{\eta_0}$$

$$\mathrm{i}\frac{\partial}{\partial t}G^{-+}(\boldsymbol{k})|_{\eta_0} = \omega_{\boldsymbol{k}}G^{-+}(\boldsymbol{k})|_{\eta_0}$$

At η_* the boundary conditions require that G^{++} and G^{-+} are equal

$$G^{++}(\boldsymbol{k})|_{\eta_0} = G^{-+}(\boldsymbol{k})|_{\eta_0}$$

Also, G^{+-} is the Hermitian conjugate of G^{-+} and G^{--} is the Hermitian conjugate of G^{++}

In the vacuum case, the equations to solve are

$$\left(\frac{\partial^2}{\partial\eta^2}+2\frac{a'}{a}\frac{\partial}{\partial\eta}+k^2+a^2m^2\right)G^{++}=-\mathrm{i}\delta(\eta-\tau)$$ G^{++} is a Green's function

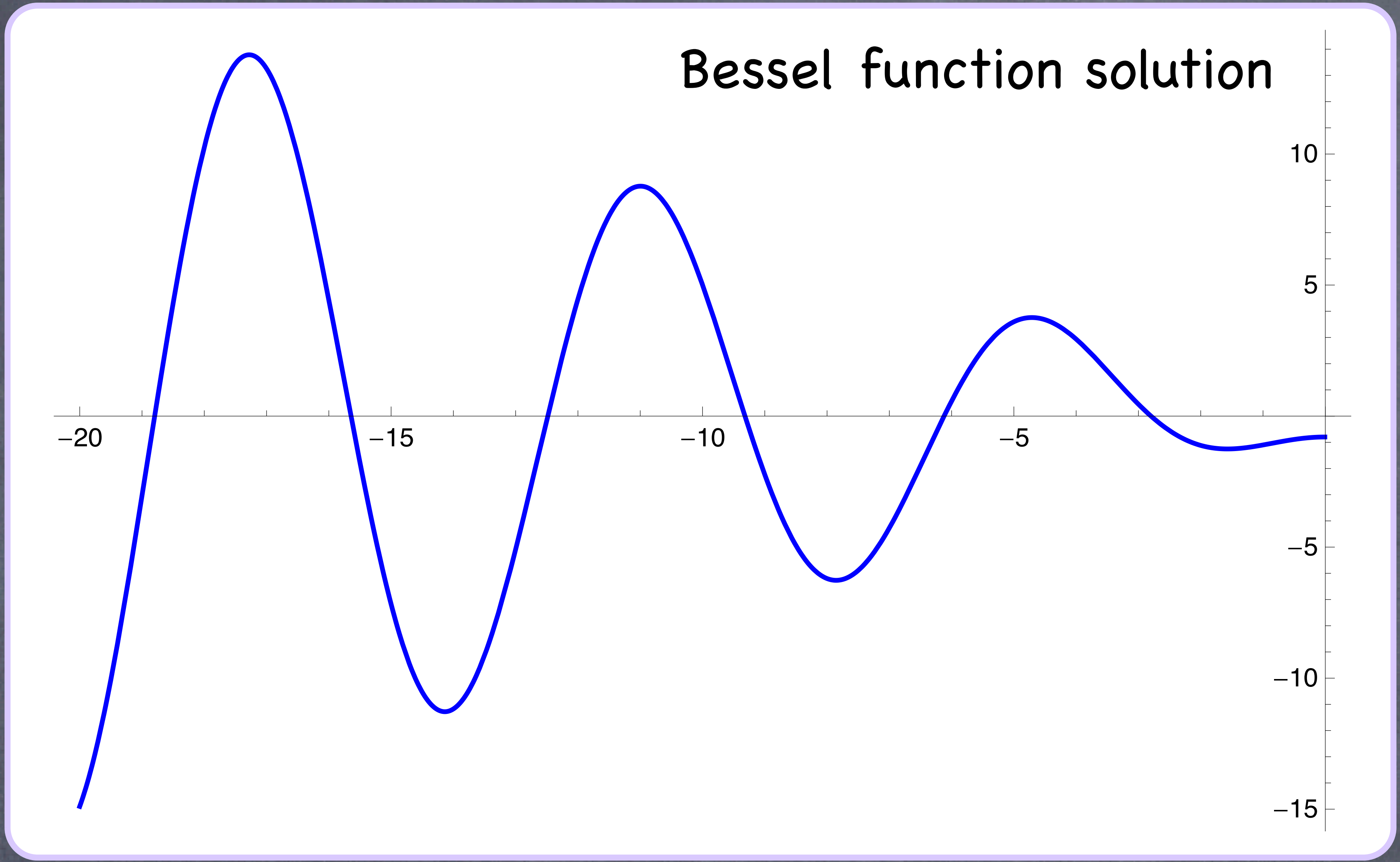
$$\left(\frac{\partial^2}{\partial\eta^2}+2\frac{a'}{a}\frac{\partial}{\partial\eta}+k^2+a^2m^2\right)G^{-+}=0$$ G^{-+} is just homogeneous

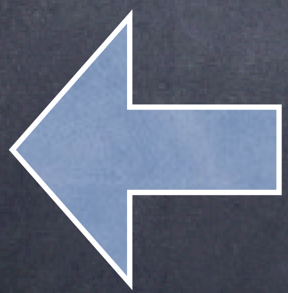
Define $x = k\eta = -k/aH$ and $G^{++} = u^{++}(-x)^{1/2}/a$

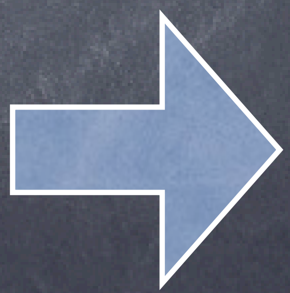
Now take H to be constant for just a few efolds around horizon crossing, where $x \approx 1$ (obviously we will have to work harder later)

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}+\left[1-\frac{9/4-m^2/H^2}{x^2}\right]\right)u^{++}=-\frac{\mathrm{i}}{a}\frac{1}{k(-x)^{1/2}}\delta(x-y)$$

Bessel equation of order $\nu^2 = 9/4 - m^2/H^2$







Approaches pure negative frequency on subhorizon scales

Approaches a constant a few e-folds after horizon-crossing

In the massless case we get a famous result

$$G^{++} = (2\pi)^3 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2) \frac{H_*^2}{2k^3} \times \begin{cases} (1 - \mathrm{i}k\eta)(1 + \mathrm{i}k\tau)\mathrm{e}^{\mathrm{i}k(\eta - \tau)} & \eta < \tau \\ (1 + \mathrm{i}k\eta)(1 - \mathrm{i}k\tau)\mathrm{e}^{-\mathrm{i}k(\eta - \tau)} & \tau < \eta \end{cases}$$

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momentum conservation

k is common value of k_1 and k_2

H_* is the nearly constant value of H during horizon exit

valid from $|k\eta| \approx \exp(+\text{few})$ to $|k\eta| \approx \exp(-\text{few})$

Also, G^{-+} is a solution of the homogeneous equation.
The bc says it agrees with G^{++} for $\eta = \eta_*$, for all values of τ,
but is positive frequency. Therefore

$$G^{-+} = (2\pi)^3 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2) \frac{H_*^2}{2k^3} (1 + \mathrm{i}k\eta)(1 - \mathrm{i}k\tau)\mathrm{e}^{\mathrm{i}k(\eta-\tau)}$$

This estimate is only valid until $|k\eta| \approx \exp(-\text{few})$, but by that time the fluctuation has settled down to a near constant

$$\langle \phi(\boldsymbol{k}_1)\phi(\boldsymbol{k}_2) \rangle = (2\pi)^3 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2) \frac{H_*^2}{2k^3}$$

Since H is changing only slowly, the amplitude depends only weakly on k

As you heard yesterday, in a single-field model, it is a theorem that the density perturbation this generates is constant outside the horizon (it decouples from the infrared dynamics).

But more generally we need to work harder.

We don't try to describe modes above the cutoff.
Maybe the modes of quantum fields aren't the right description.

CUTOFF

Presumably some fluctuations which are heavy compared to the Hubble scale

Hubble scale - energy density of the background

At least one fluctuation which is light compared to the Hubble scale

A mode with fixed comoving wavenumber k begins life far above the cutoff, where we are clueless

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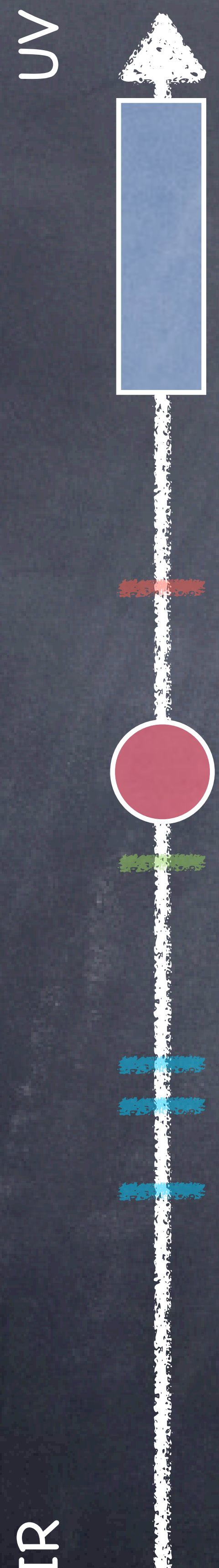

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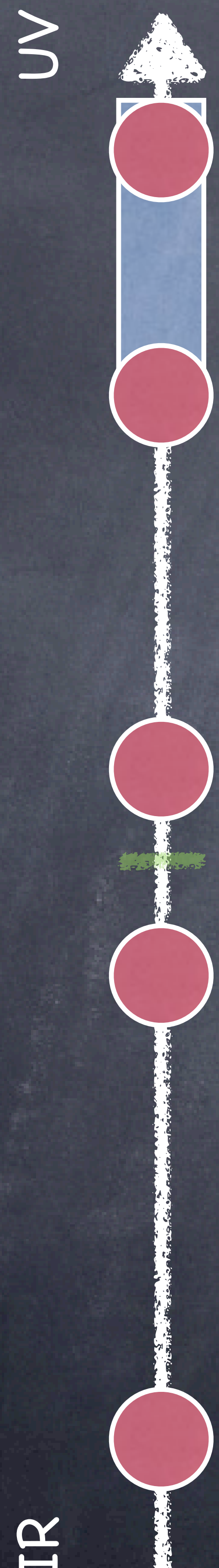
If it is one of the light modes, it can continue to have dynamics deep in the IR



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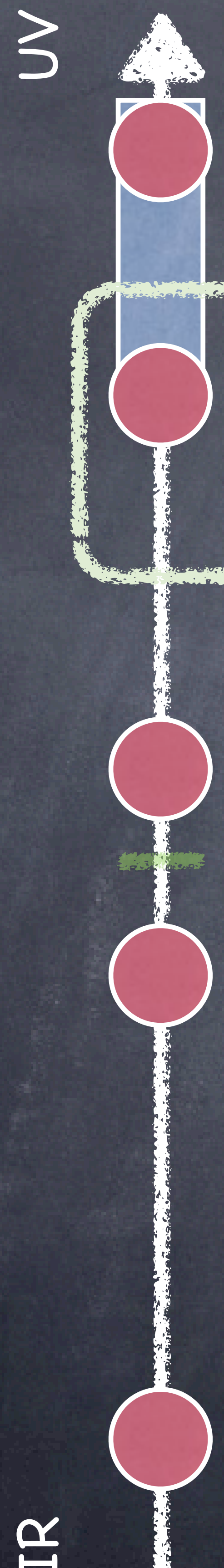

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In principle, this is what the density matrix ICs do

CUTOFF

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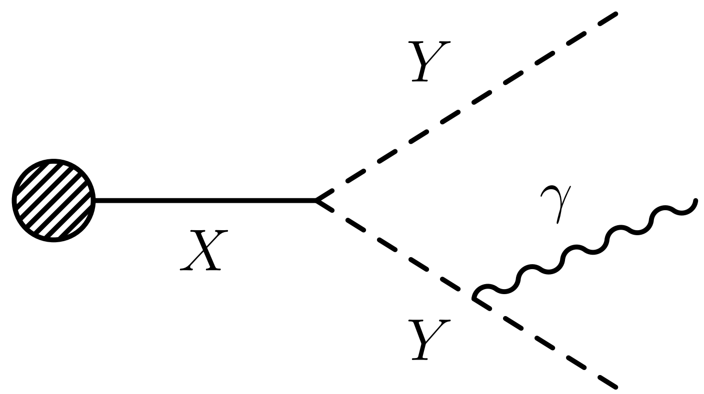
So, when we do the standard calculation, we are
not assuming that we know physics above the cutoff
even though the mode begins far, far above it

However, we certainly are assuming something

If we use vacuum bcs, then we are assuming that whatever the high energy physics is, it generates modes in their vacuum when they join the field theory description.

It could not be like that. Then we would have some mixture of positive and negative frequency modes.
It turns out this has consequences for the 3pf.

Usually, people emphasize the similarity of (unfamiliar) in-out to (familiar) in-in



"New source" of gravitational waves
à la Senatore, Silverstein & Zaldarriaga (1109.0542)

This diagram is what we would compute to obtain the decay rate

What should we do for in-in?

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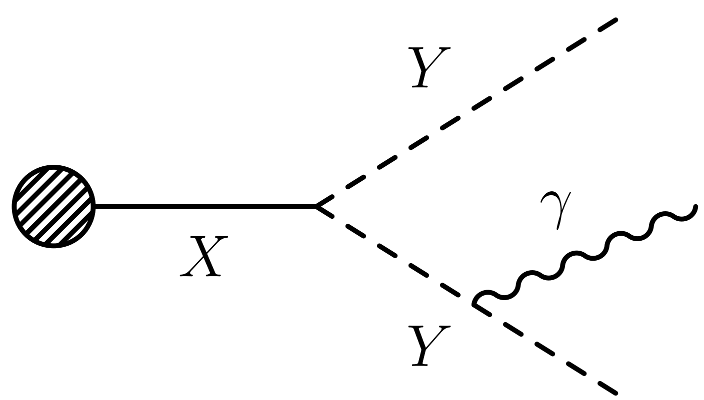

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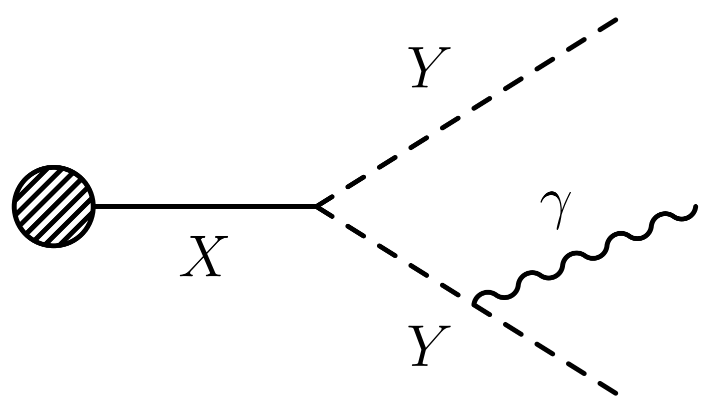

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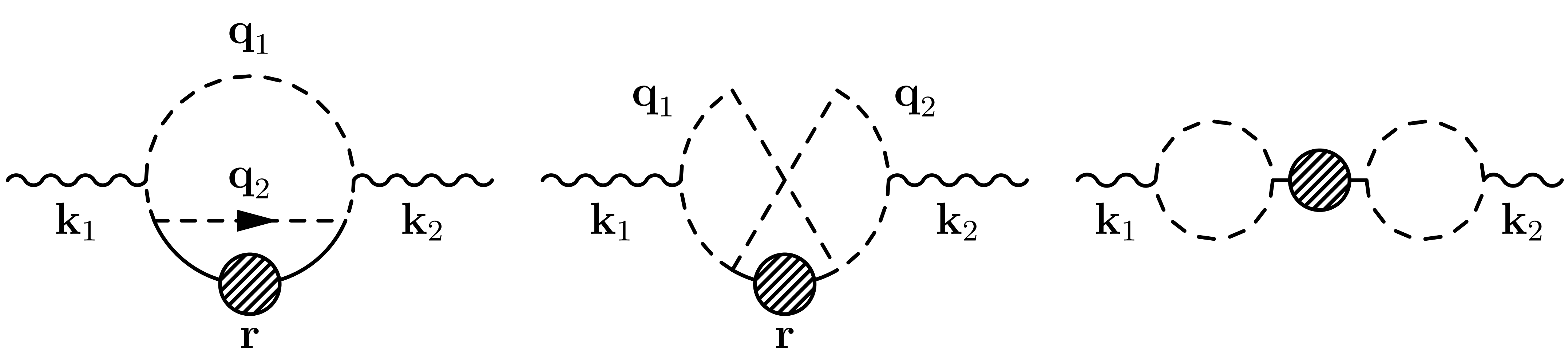

There are three diagrams, and they are not trees

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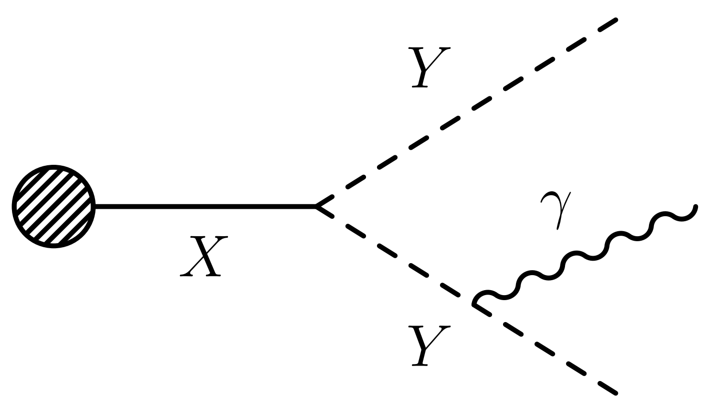

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Wednesday, 18 July 12

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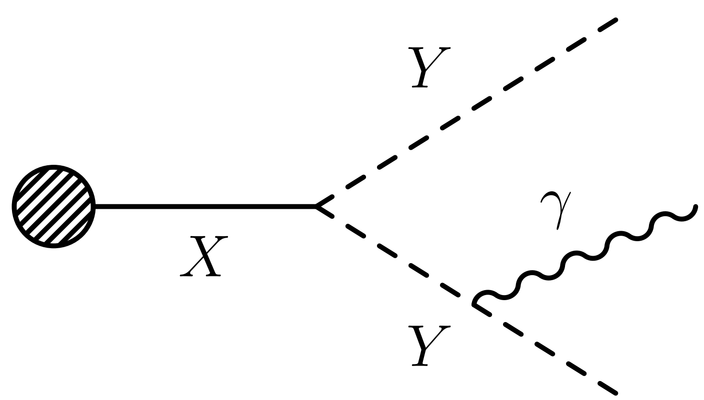

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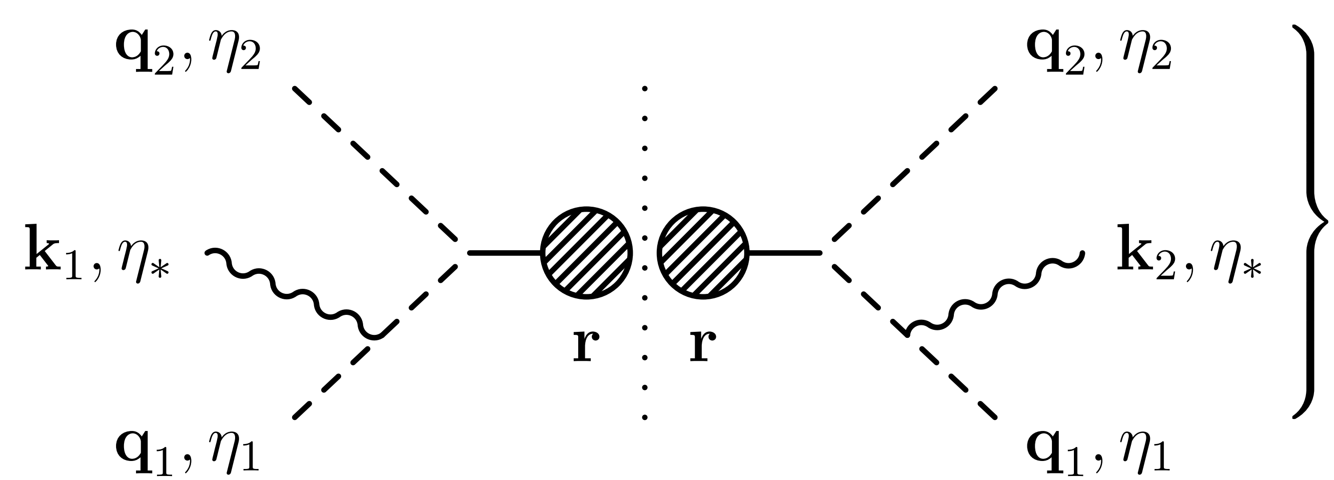
We get them by sewing together two copies of the decay diagram and averaging over unobserved particles

$$\int \frac{\mathrm{d}^3 r}{(2\pi)^3} \frac{\mathrm{d}^3 q_1}{(2\pi)^3} \frac{\mathrm{d}^3 q_2}{(2\pi)^3} \mathrm{d}\eta_1 \, \mathrm{d}\eta_2 \left\{ \quad \right\}$$



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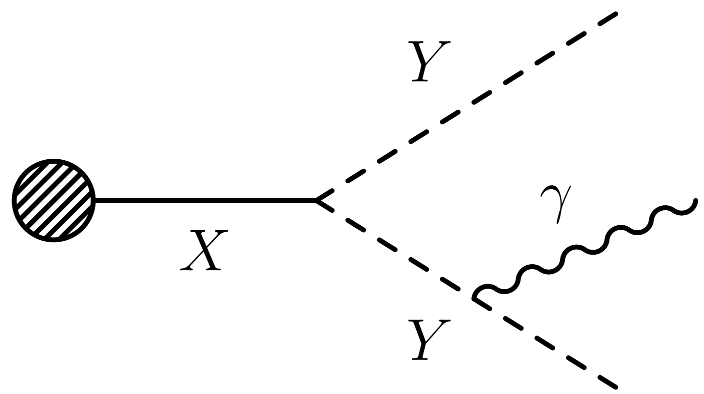

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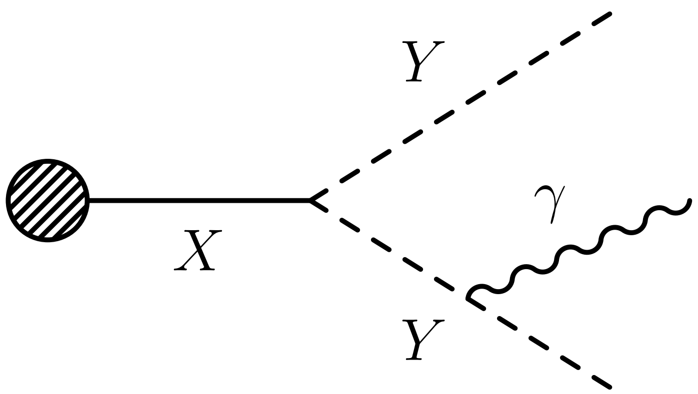

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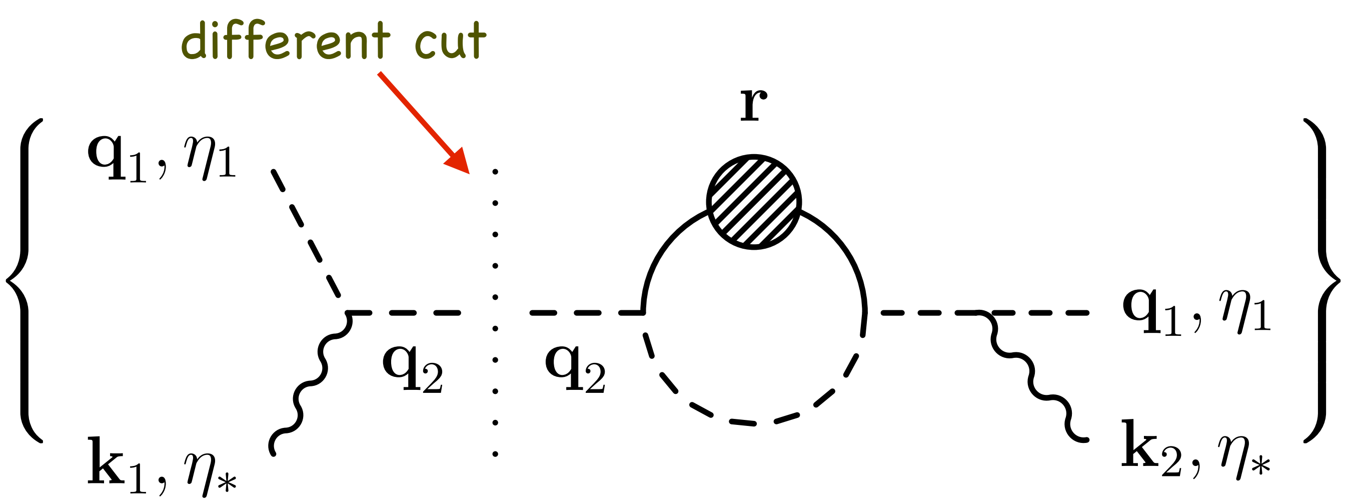
$$\int \frac{\mathrm{d}^3 q_1}{(2\pi)^3} \frac{\mathrm{d}^3 q_2}{(2\pi)^3} \mathrm{d}\eta_1$$



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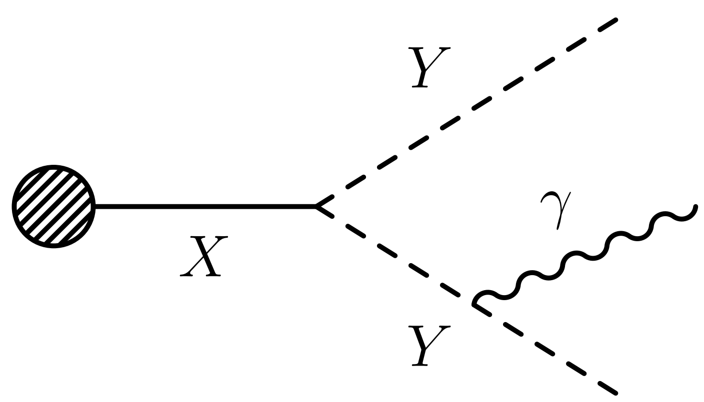

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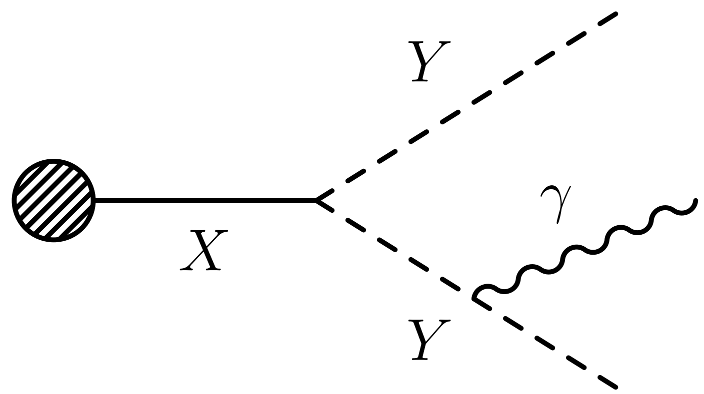

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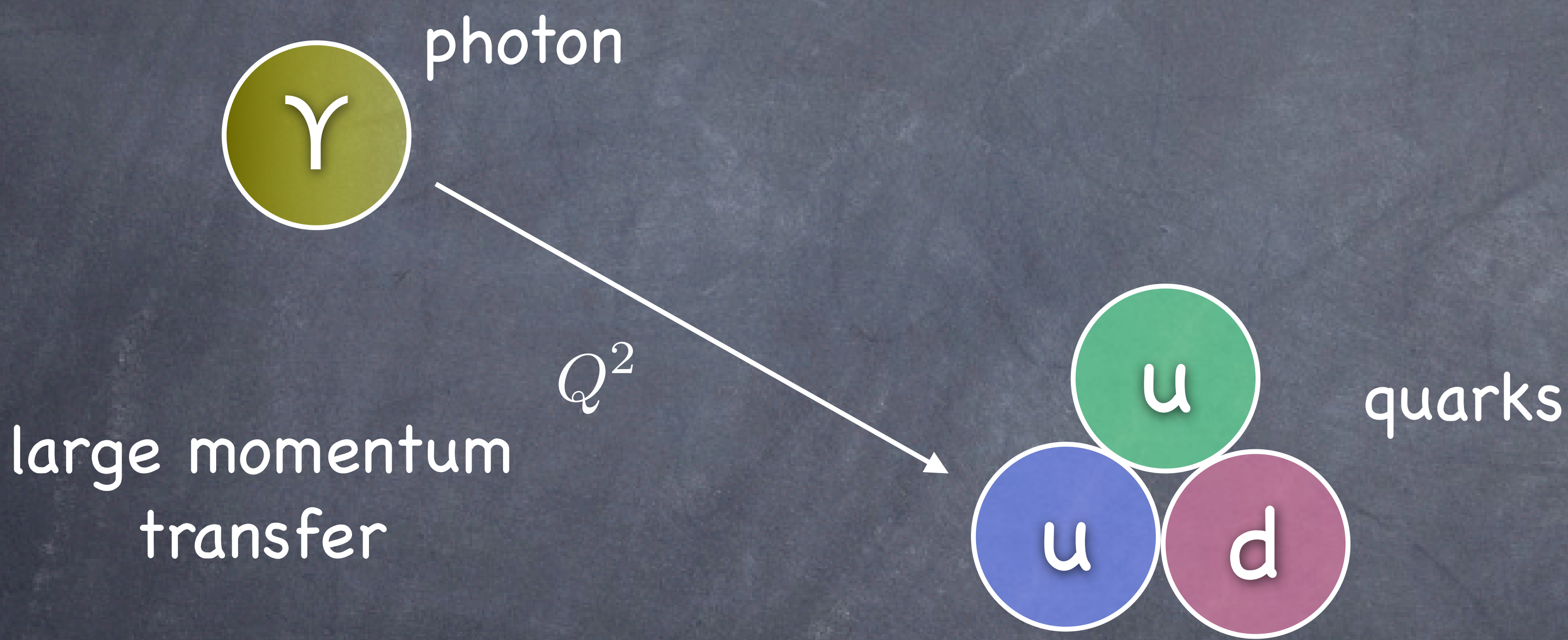
The moral is that an in-in calculation sums over: (1) all possible final state particles, and (2) all possible ways that these can appear in the final state, including interference effects when we go from amplitudes to probabilities.

In does this in a very economical way, at the cost of some ambiguity in interpretation of loop diagrams.

We would like to observe the presence of intermediate states (heavy or light) — and if possible in a relatively unambiguous way

This brings us very close to something like QCD, where we would like to observe the presence of quarks



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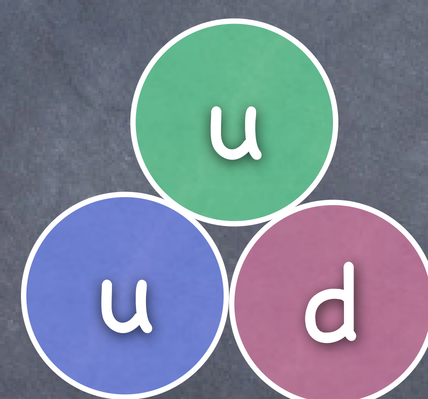

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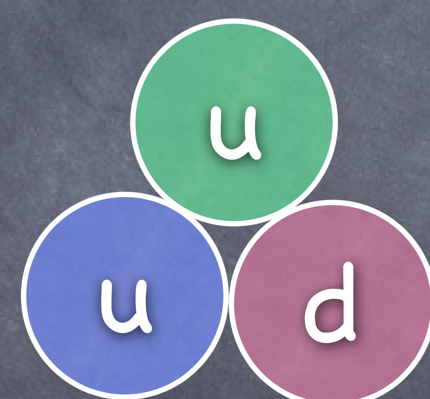

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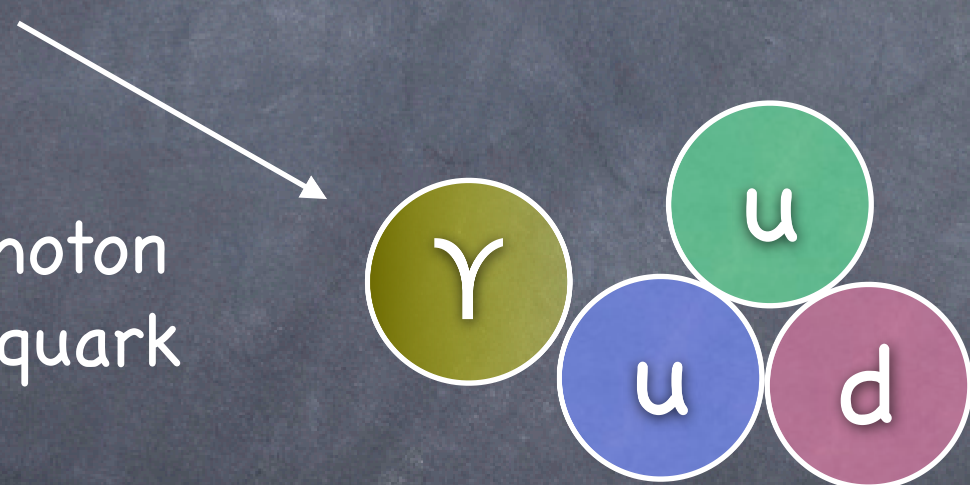
impinging photon strikes one quark



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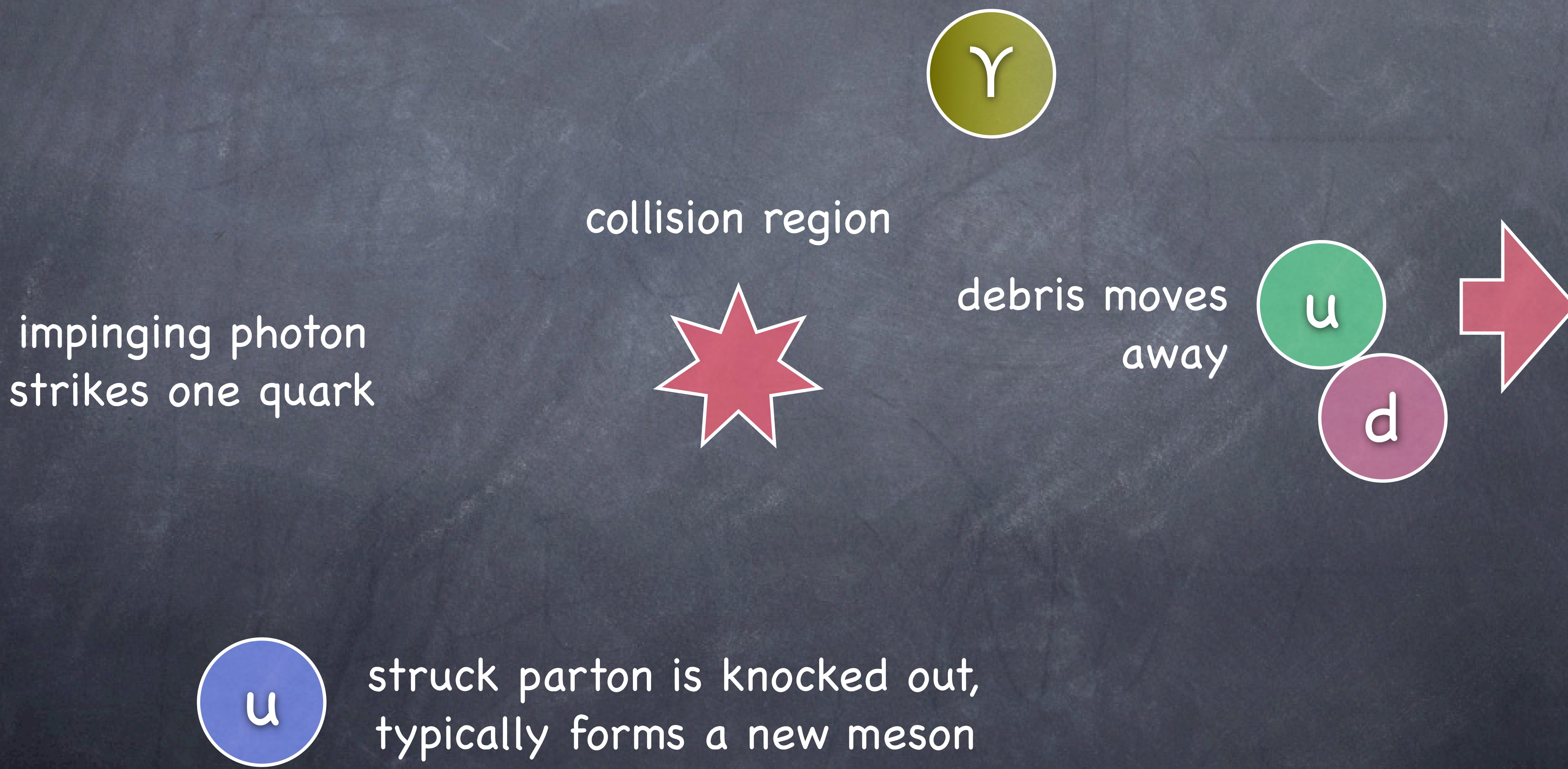

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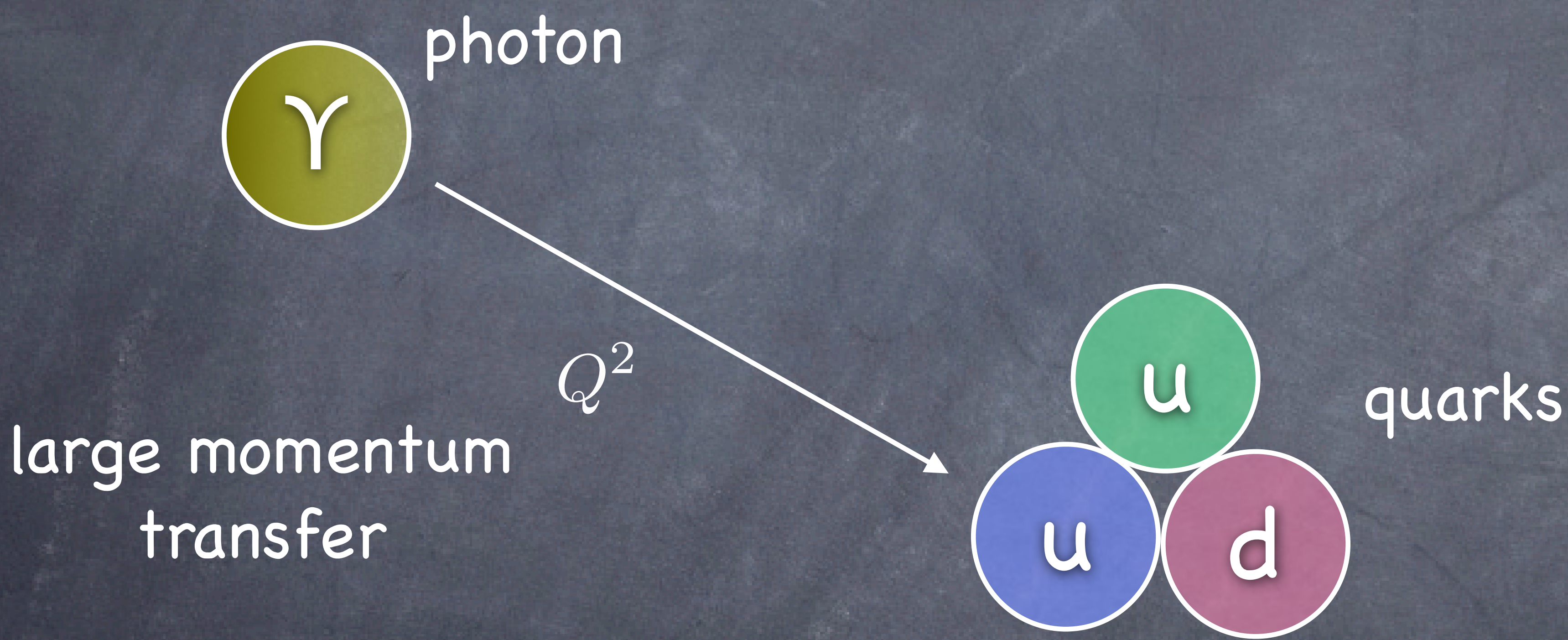

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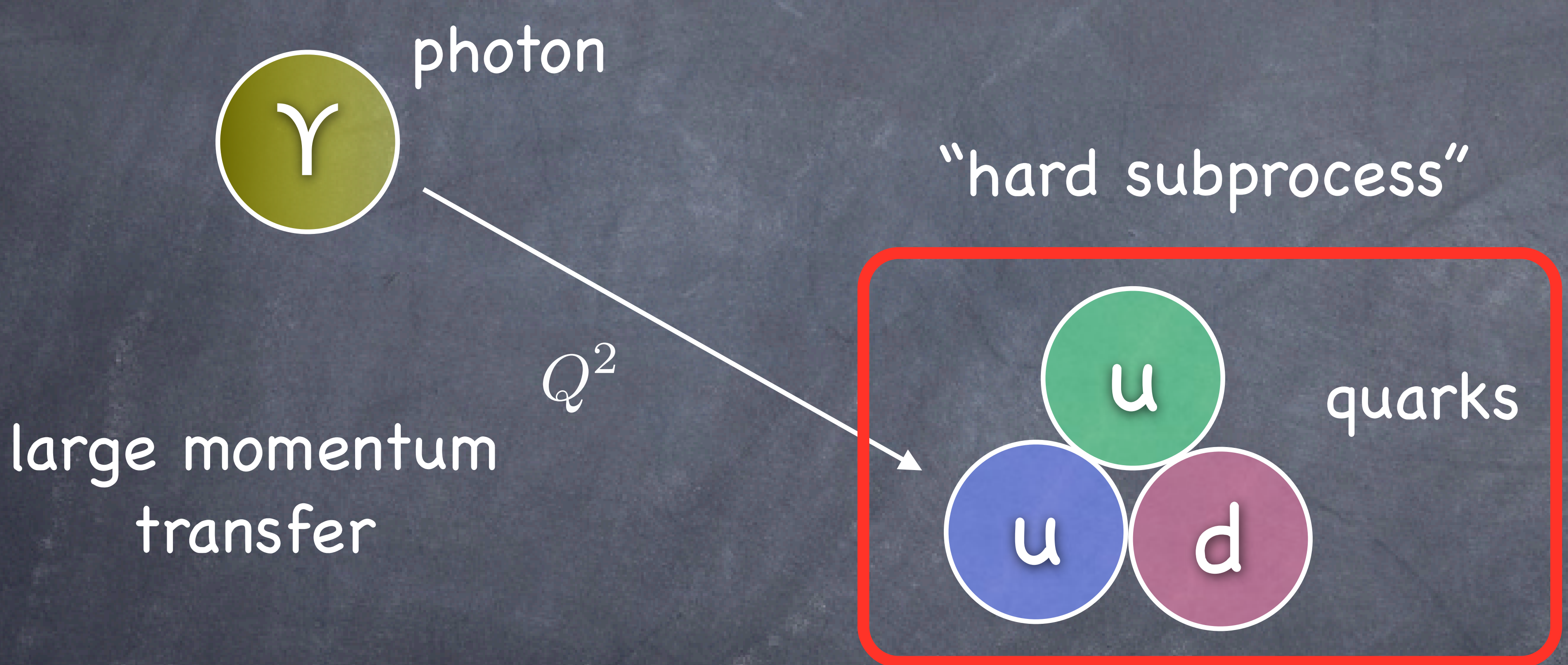

collision takes place rapidly, at high energy E_{hard}, where QCD coupling is small

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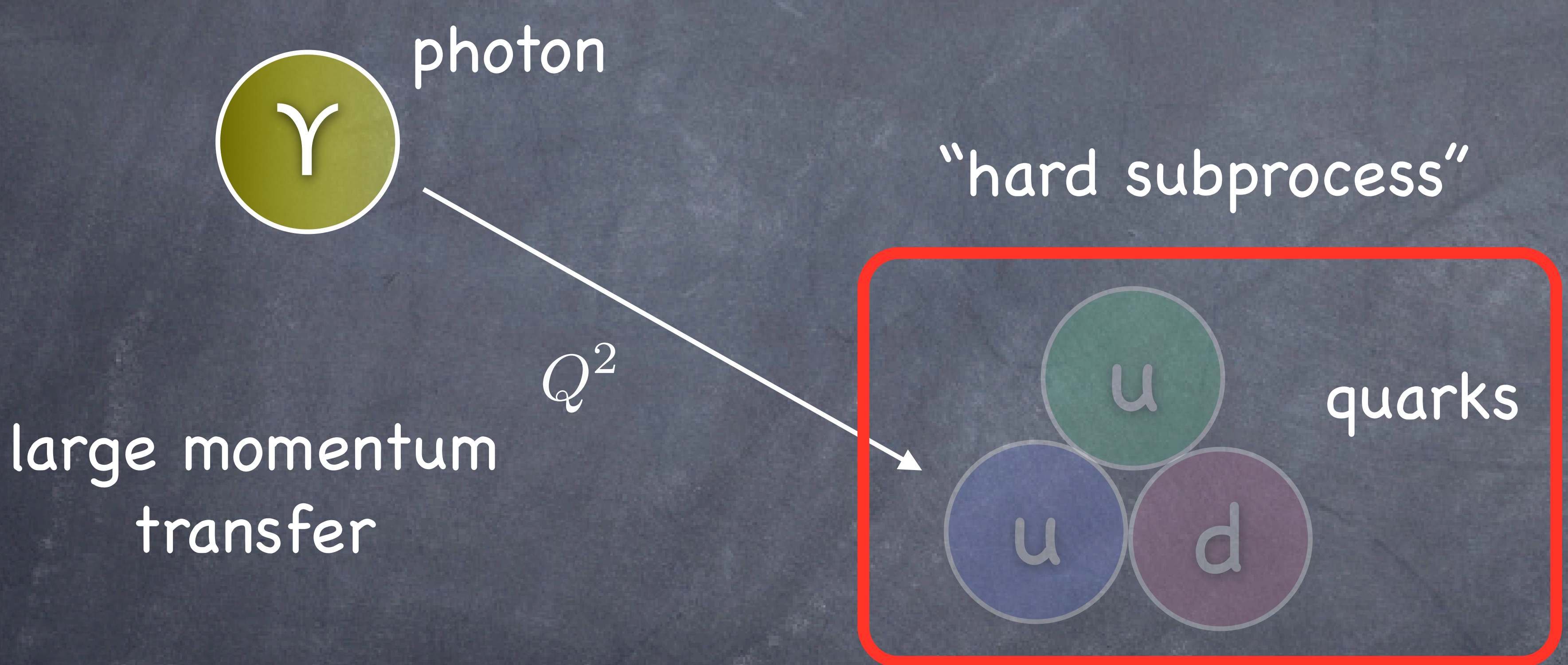

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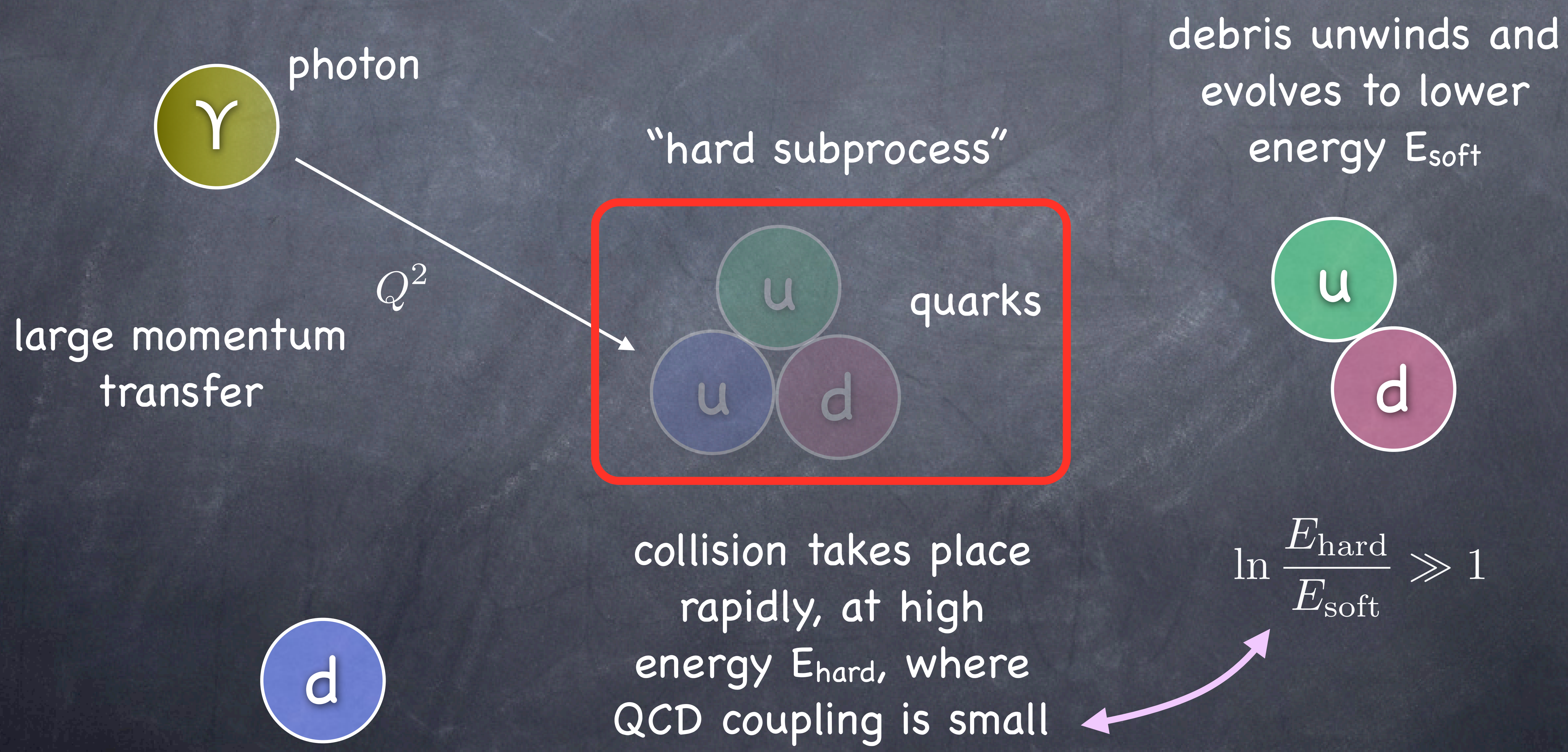
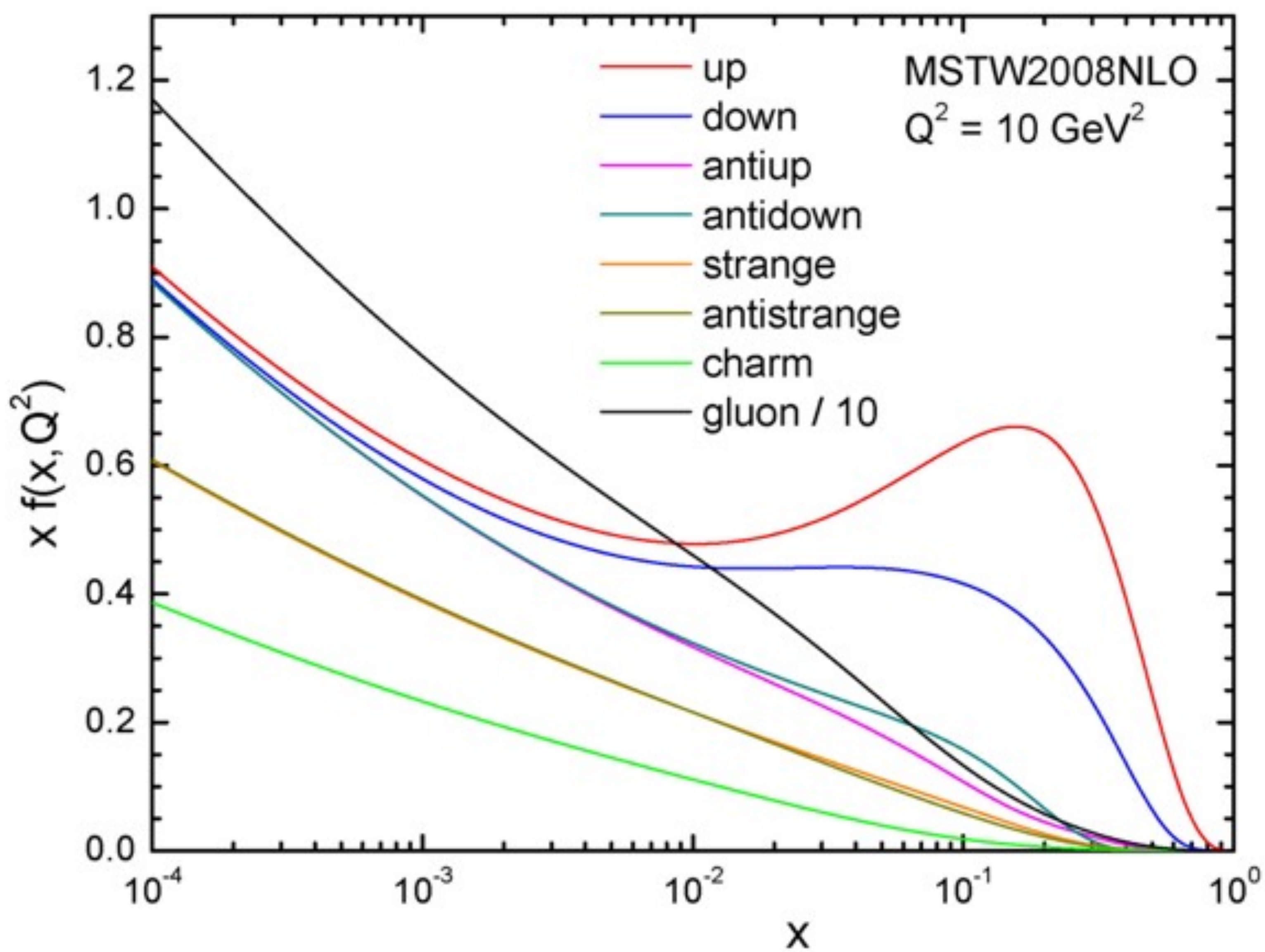


The hard subprocess is essentially just scattering of solid spheres. There's not much diagnostic here.

$$\ln \frac{E_{\text{hard}}}{E_{\text{soft}}} \gg 1$$

Instead, details of the theory show up in these large logs. But it's no good just calculating to a few more orders in PT.



Credit: James Stirling

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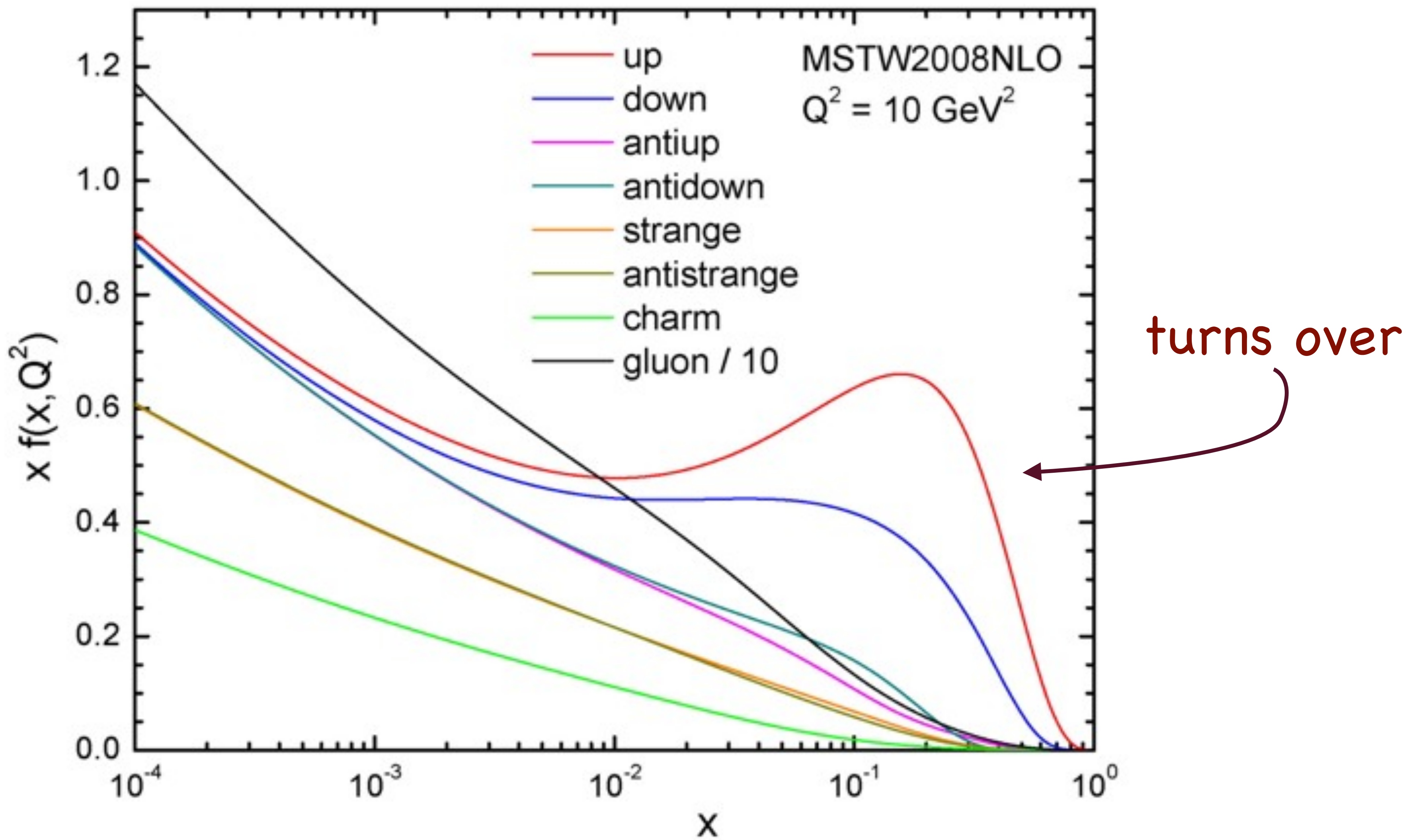

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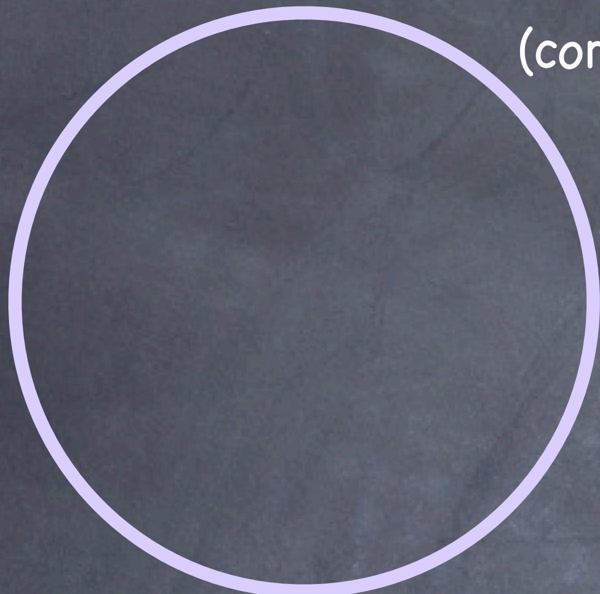
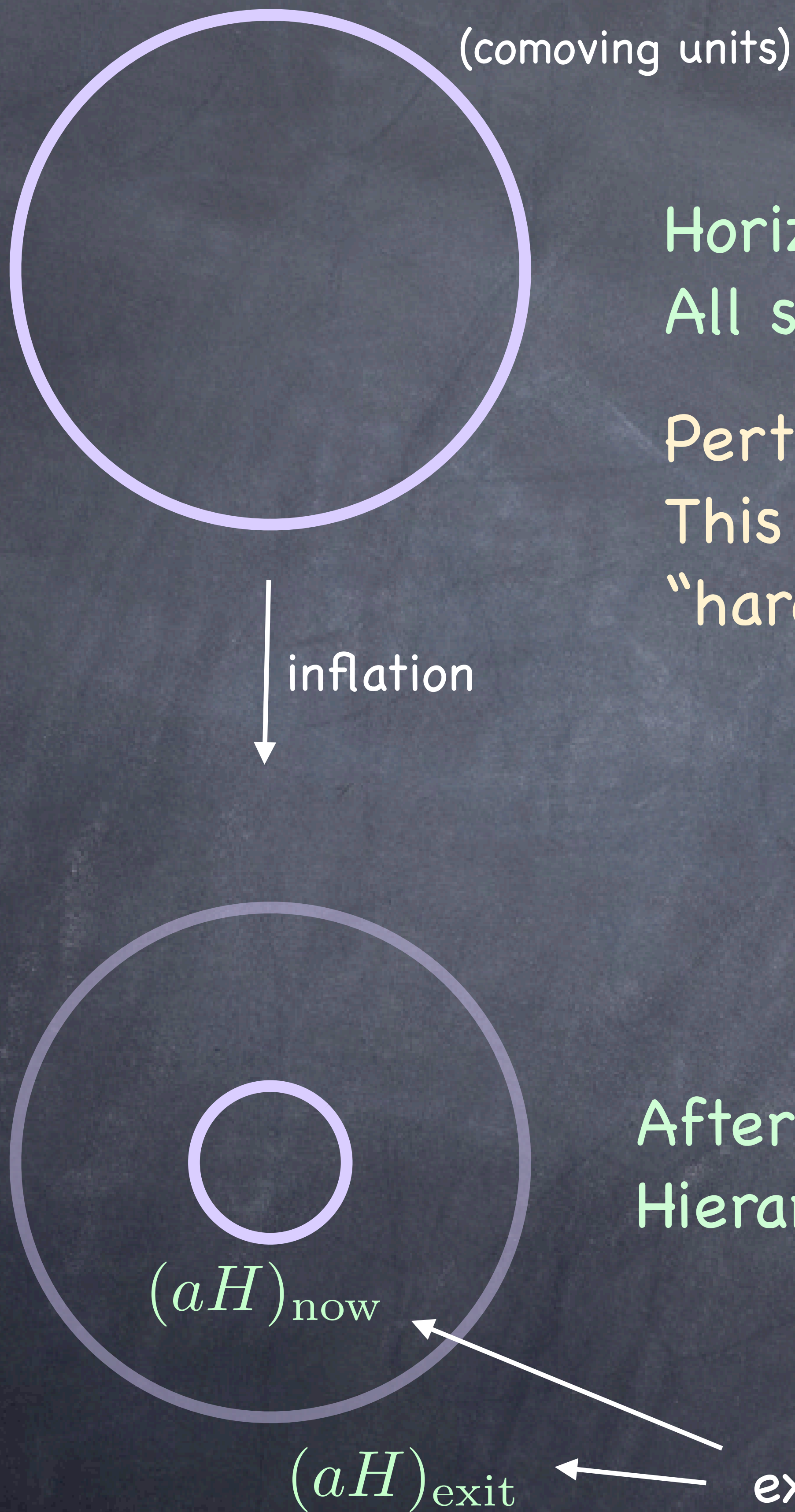


Horizon exit:
All scales comparable $aH \sim k_i \sim k_*$

Perturbation theory is acceptable.
This is a very close analogue of the "hard subprocess" in pQCD



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After horizon exit:
Hierarchy of scales $\ln \dfrac{(aH)_{\rm exit}}{(aH)_{\rm now}} = \ln |k_{\rm exit}\eta| \gg 1$

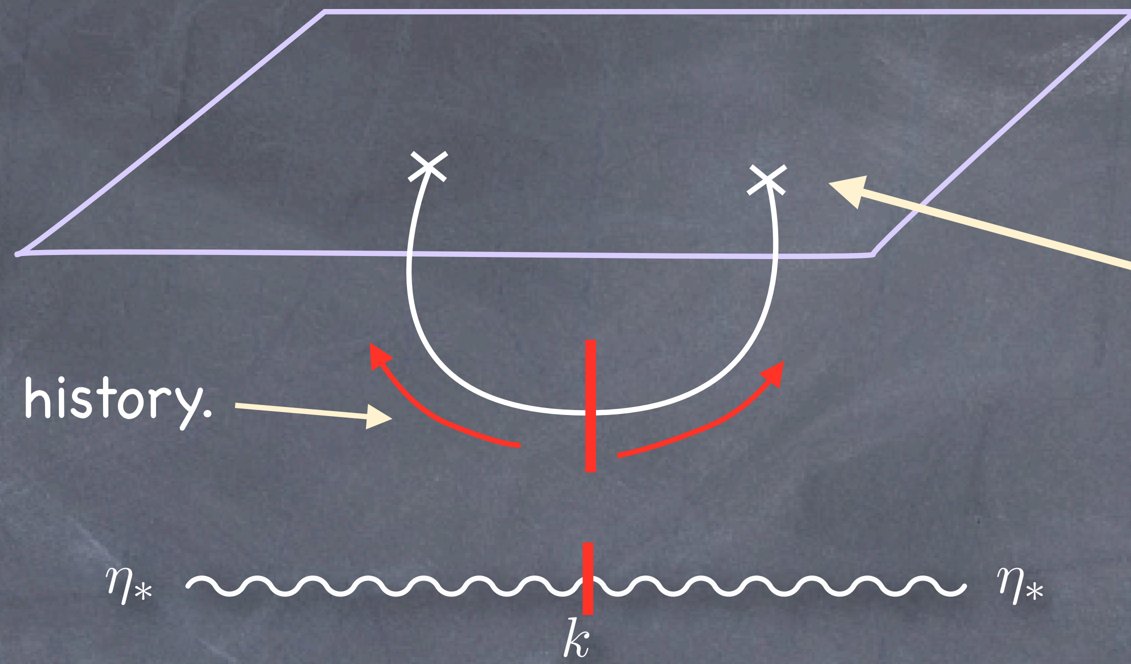
late time, fixed state

η_2

two quanta appear and then separate, sharing a history. So, they are correlated.

early time, fixed state

η_1



"Schwinger" formulation

both external legs at late time, so no quanta enter the diagram

instead, they are nucleated like an instanton

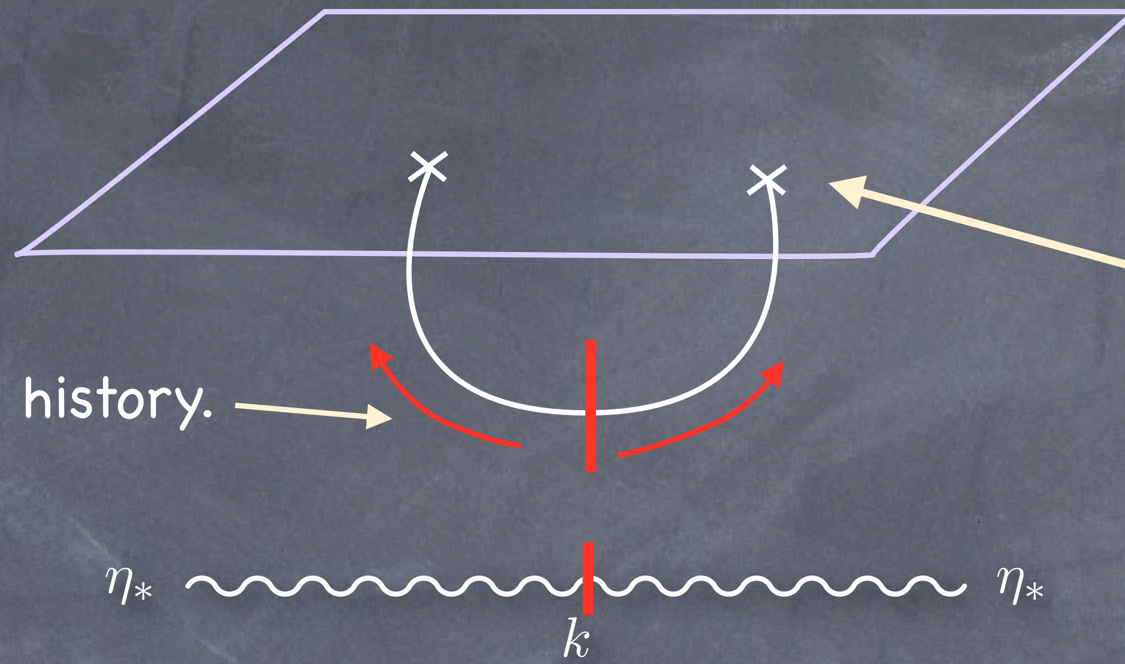
~~late time, fixed state~~

~~η_2~~

two quanta appear and then separate, sharing a history. So, they are correlated.

early time, fixed state

η_1

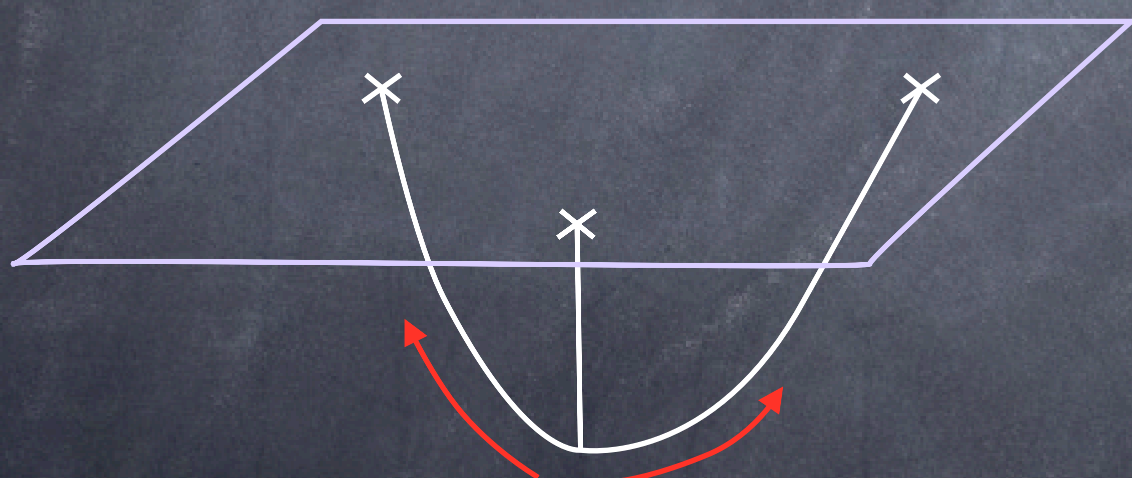


"Schwinger" formulation

both external legs at late time, so no quanta enter the diagram

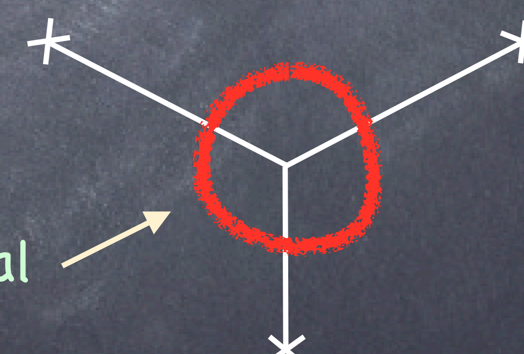
instead, they are nucleated like an instanton

precisely the same thing happens for, eg., the 3pf



3 quanta nucleate and separate

the Feynman rules always give an integral over all space



$$\int \mathrm{d}^4x \sqrt{-g} \cdots$$

$$\mathrm{d}^3x \, \mathrm{d}t \; a(t)^3$$

This divergence, and loops, give different species of logarithm
These all depend on the infrared dynamics of the theory

$\ln |k\eta_*|$ Time-dependence

$\ln \frac{k}{k_*}$ Scale-dependence.

$\ln kL$ Depend on the tile size we chose at the outset. This wasn't physical; they have no meaning by themselves, but only as a proxy for something else.

$\ln \frac{k_i}{k_t}$ Also occur and can be thought of as an infrared effect of a different type. In an n-point function, these depend on the shape of the momentum n-gon. Become large when $k_i/k_t \ll 1$, ie., the "squeezed limit". [coming later]

time scales
(slow roll scales)

$$\epsilon \sim \frac{V'^2}{V^2} \quad \eta \sim \frac{V''}{V} \quad \xi \sim \frac{V'''V'}{V^2}$$

$\mathbf{10^{-2}}$

quantum scale

$$\frac{H^2}{M_{\mathrm{P}}^2}$$

$\mathbf{10^{-10}}$ ish

This divergence at late times produces a logarithm in the 3pf, associated with one of the slow-roll time scales

$$\langle \delta\phi(\boldsymbol{k}_1)\phi(\boldsymbol{k}_2)\phi(\boldsymbol{k}_3)\rangle_* \supseteq (2\pi)^3\delta(\sum_i \boldsymbol{k}_i)\frac{H_k^2 V_k'''}{12\prod_i k_i^3}(N_* - N_k)\sum_i k_i^3$$

Falk, Rangarajan & Srednicki (1992)

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Falk, Rangarajan & Srednicki (1992)

time scale, will become ξ on translation to the curvature perturbation

number of e-folds outside the horizon, grows to between 40 and 60 during observable inflation

Sasaki, Suzuki, Yamamoto & Yokoyama (1993) "Superexpansionary" divergence — a geometrical effect associated with the growing volume of space available at very late times

Different sources of time dependence

Associated with the slow-roll time scale

Arise from higher-order slow-roll corrections

Describe evolution of correlations outside the horizon, which can be understood using a classical phase space picture. We already have to work to all orders.

Associated with the quantum scale H^2/M_P^2

Arise from loops

Probably become important on a time scale of order M_P^2/H^2 efolds. They are quantum corrections to the time evolution, but the huge time scale makes them mostly irrelevant for observable inflation. Could be important for a quantitative description of eternal inflation.

To next-order in powers of slow-roll, the two-point function is (now for multiple fields, labelled by α, β, ...)

$$\langle \delta\phi_\alpha(\mathbf{k}_1)\delta\phi_\beta(\mathbf{k}_2)\rangle_\eta \supseteq (2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2)\frac{H_*^2}{2k^3} \times \left\{ \delta_{\alpha\beta}\left[1+2\epsilon_*\left(1-\gamma_{\mathrm{E}}-\ln\frac{2k}{k_*}\right)\right] + 2u_{\alpha\beta}^*\left[2-\ln(-k_*\eta)-\ln\frac{2k}{k_*}-\gamma_{\mathrm{E}}\right]\right\}$$

$$u_{\alpha\beta} = -\frac{m_{\alpha\beta}}{3H^2}$$

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Structurally, we expect each order in slow-roll to be proportional to $1/k^3$, by scale invariance

$$\langle \delta\phi_\alpha(\mathbf{k}_1)\delta\phi_\beta(\mathbf{k}_2)\rangle_\eta = (2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2)\frac{\Sigma_{\alpha\beta}}{2k^3}$$

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This procedure is one way to think about the renormalization group - it is just inversion of a Taylor expansion!

For example, expand a function A around an arbitrary point x* (just asymptotic - need not be convergent)

$$A(x) = A_* \left[1 + \beta_*(x - x_*) + \cdots\right]$$

This tells us two things:

$$\left.\frac{\mathrm{d}A}{\mathrm{d}x}\right|_{x=x_*} = A_* \beta_*$$

and

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But since this is true for any x_*

$$\frac{\mathrm{d}\ln A(x)}{\mathrm{d}x} = \beta(x)$$

The zero-order term gives an ic

In our case, we have a matrix Taylor expansion, so we have to be careful with the indices

$$\frac{\mathrm{d}\Sigma_{\alpha\beta}}{\mathrm{d}N} = u_{\alpha\gamma}\Sigma_{\gamma\beta} + u_{\beta\gamma}\Sigma_{\gamma\alpha}$$

and the initial condition can be extracted from the zero-order term

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$$\Sigma_{\alpha\beta} = H_*^2\delta_{\alpha\beta}$$ (at horizon crossing)

If you have seen the Boltzmann equation before, you know this can be solved using an integrating factor

$$\Sigma_{\alpha\beta} = \Gamma_{\alpha i}\Gamma_{\beta j}S_i j$$

$$\left(\frac{\mathrm{d}\Gamma_{\alpha i}}{\mathrm{d}N} - u_{\alpha\gamma}\Gamma_{\gamma i}\right)\Gamma_{\beta j}S_{ij} + \left(\frac{\mathrm{d}\Gamma_{\beta j}}{\mathrm{d}N} - u_{\beta\gamma}\Gamma_{\gamma j}\right)\Gamma_{\alpha i}S_{ij} + \Gamma_{\alpha i}\Gamma_{\beta j}\frac{\mathrm{d}S_{ij}}{\mathrm{d}N} = 0$$

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set this equal to zero ⟶ $$\frac{\mathrm{d}\Gamma_{\alpha i}}{\mathrm{d}N} = u_{\alpha\gamma}\Gamma_{\gamma i}$$

This has a formal solution in terms of a path-ordered exponential

$$\Gamma_{\alpha i} = \mathrm{P}\exp\left(\int_{N_0}^{N} \mathrm{d}N' \, \boldsymbol{u}\right)_{\alpha i}$$

(But it is not often directly useful)

Here, I have set the initial condition to be

$$\Gamma_{\alpha i} = \delta_{\alpha i}$$

at the initial time

$$\left(\frac{\mathrm{d}\Gamma_{\alpha i}}{\mathrm{d}N} - u_{\alpha\gamma}\Gamma_{\gamma i}\right)\Gamma_{\beta j}S_{ij} + \left(\frac{\mathrm{d}\Gamma_{\beta j}}{\mathrm{d}N} - u_{\beta\gamma}\Gamma_{\gamma j}\right)\Gamma_{\alpha i}S_{ij} + \Gamma_{\alpha i}\Gamma_{\beta j}\frac{\mathrm{d}S_{ij}}{\mathrm{d}N} = 0$$

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Each inflationary trajectory is traced out by the equation

$$\frac{\mathrm{d}\phi_\alpha}{\mathrm{d}N} = -\frac{V_{,\alpha}}{3H^2} = u_\alpha$$

$$\left(\frac{\mathrm{d}\Gamma_{\alpha i}}{\mathrm{d}N} - u_{\alpha\gamma}\Gamma_{\gamma i}\right)\Gamma_{\beta j}S_{ij} + \left(\frac{\mathrm{d}\Gamma_{\beta j}}{\mathrm{d}N} - u_{\beta\gamma}\Gamma_{\gamma j}\right)\Gamma_{\alpha i}S_{ij} + \Gamma_{\alpha i}\Gamma_{\beta j}\frac{\mathrm{d}S_{ij}}{\mathrm{d}N} = 0$$

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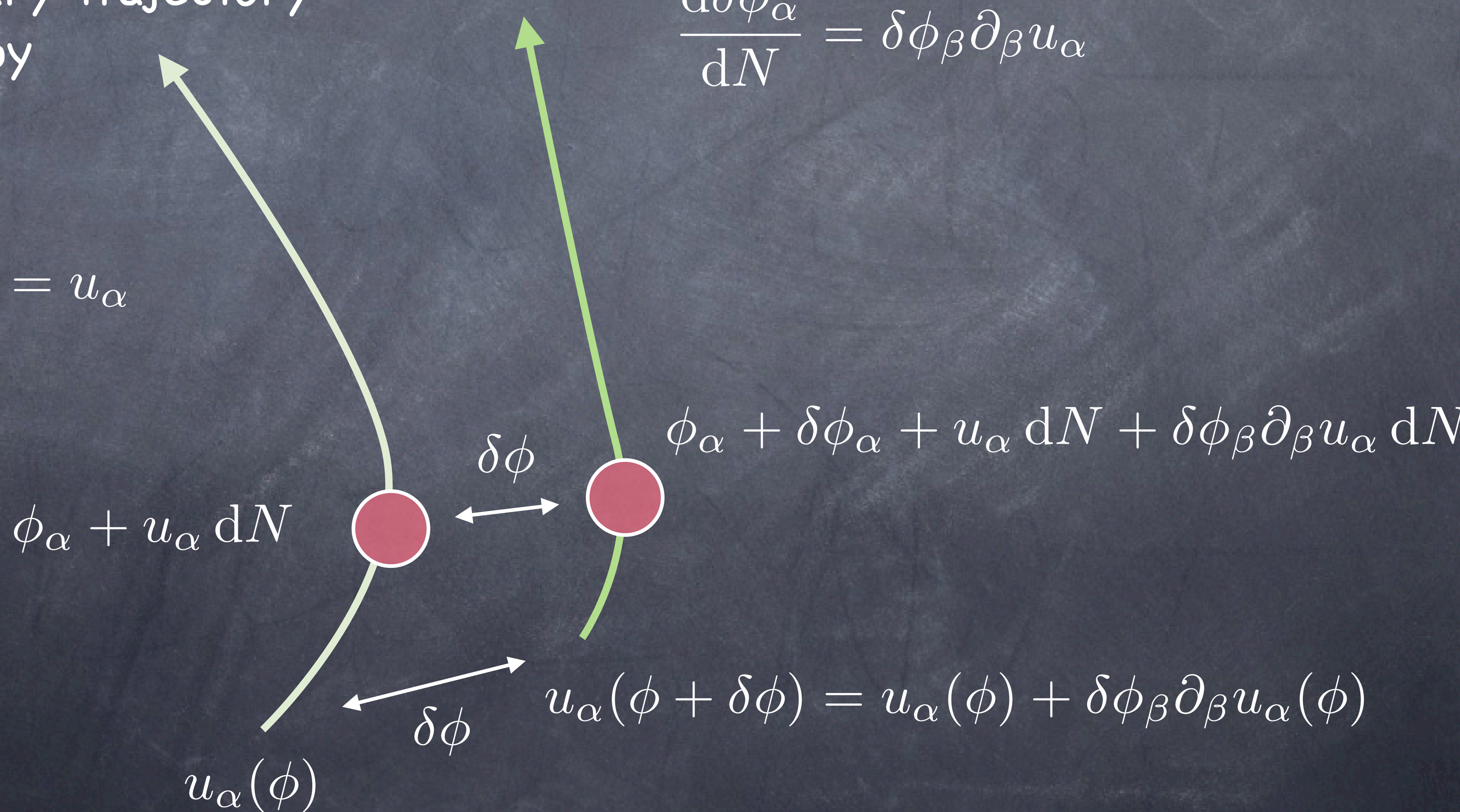

$$u_\alpha(\phi + \delta\phi) = u_\alpha(\phi) + \delta\phi_\beta\partial_\beta u_\alpha(\phi)$$

$$\left(\frac{\mathrm{d}\Gamma_{\alpha i}}{\mathrm{d}N} - u_{\alpha\gamma}\Gamma_{\gamma i}\right)\Gamma_{\beta j}S_{ij} + \left(\frac{\mathrm{d}\Gamma_{\beta j}}{\mathrm{d}N} - u_{\beta\gamma}\Gamma_{\gamma j}\right)\Gamma_{\alpha i}S_{ij} + \Gamma_{\alpha i}\Gamma_{\beta j}\frac{\mathrm{d}S_{ij}}{\mathrm{d}N} = 0$$

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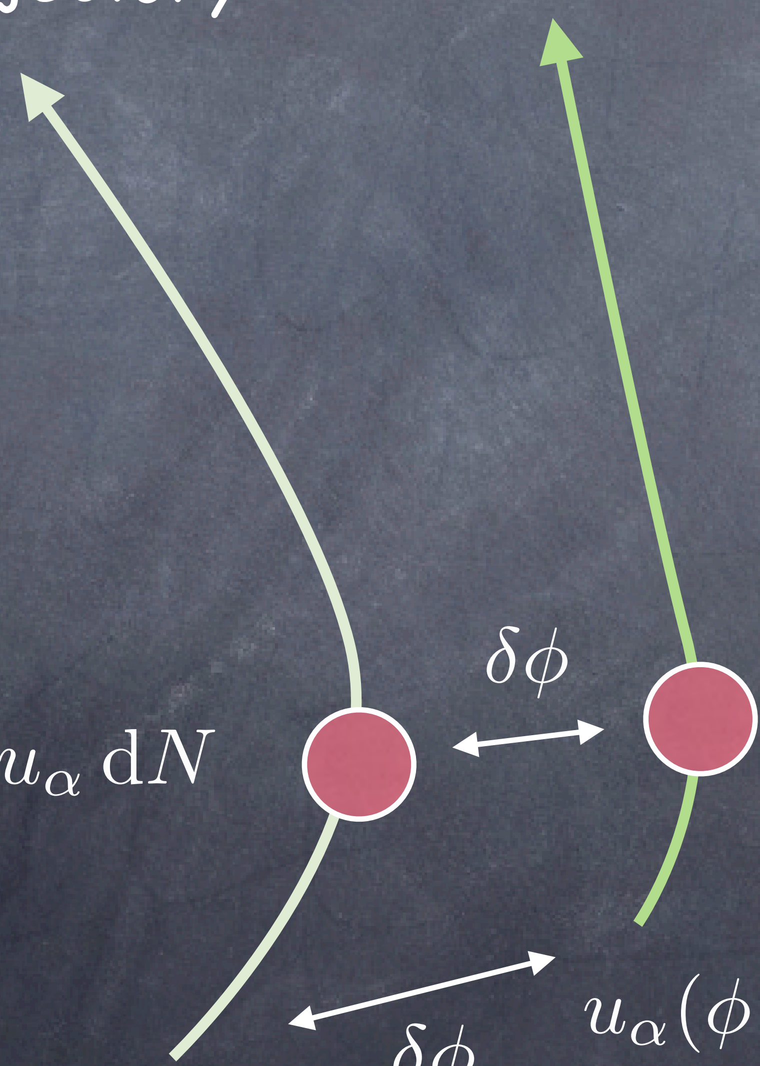
the same $u_{\alpha\beta}$



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From this we learn something very important.

If we solve with an integrating factor, then

$$\delta\phi_\alpha = \Gamma_{\alpha i}\delta_i$$

$$\left(\frac{\mathrm{d}\Gamma_{\alpha i}}{\mathrm{d}N} - u_{\alpha\gamma}\Gamma_{\gamma i}\right)\delta_i + \Gamma_{\alpha i}\frac{\mathrm{d}\delta_i}{\mathrm{d}N} = 0$$

this is already zero

chose δ_i to be constant

$$\delta\phi_\alpha(\mathrm{now}) = \Gamma_{\alpha i}\delta\phi_i(\mathrm{then})$$

so Γ is a derivative

$$\frac{\partial\phi_\alpha(\mathrm{now})}{\partial\phi_i(\mathrm{then})} = \Gamma_{\alpha i}$$

$$\left(\frac{\mathrm{d}\Gamma_{\alpha i}}{\mathrm{d}N} - u_{\alpha\gamma}\Gamma_{\gamma i}\right)\Gamma_{\beta j}S_{ij} + \left(\frac{\mathrm{d}\Gamma_{\beta j}}{\mathrm{d}N} - u_{\beta\gamma}\Gamma_{\gamma j}\right)\Gamma_{\alpha i}S_{ij} + \Gamma_{\alpha i}\Gamma_{\beta j}\frac{\mathrm{d}S_{ij}}{\mathrm{d}N} = 0$$

0 0

both these terms are zero

so this term should be zero too

Since $\Sigma_{\alpha\beta} = \Gamma_{\alpha i}\Gamma_{\beta j}S_{ij}$ we have to choose S_{ij} to be the initial value of the 2pf

Now we can finally work out what happens to the 2pf long after horizon crossing

$$\langle \delta\phi_\alpha(\boldsymbol{k}_1)\delta\phi_\beta(\boldsymbol{k}_2)\rangle_{\text{now}} = \Gamma_{\alpha i}\Gamma_{\beta j}\langle \delta\phi_i(\boldsymbol{k}_1)\delta\phi_j(\boldsymbol{k}_2)\rangle_{\text{then}}$$

$$\langle \delta\phi_\alpha\delta\phi_\beta\rangle_{\text{now}} = \frac{\partial\phi_\alpha(\text{now})}{\partial\phi_i(\text{then})}\frac{\partial\phi_\beta(\text{now})}{\partial\phi_j(\text{then})}\langle \delta\phi_i\delta\phi_j\rangle_{\text{then}}$$

If you follow the renormalization group argument for higher n-pfs, you find this pattern is reproduced at higher order

$$\delta\phi_\alpha(\text{now}) = \frac{\partial\phi_\alpha(\text{now})}{\partial\phi_i(\text{then})}\delta\phi_i(\text{then}) + \frac{1}{2}\frac{\partial^2\phi_\alpha(\text{now})}{\partial\phi_i(\text{then})\partial\phi_j(\text{then})}\delta\phi_i(\text{then})\delta\phi_j(\text{then}) + \cdots$$

This is called the "separate universe approximation/picture/expansion". It is the most common way to do analytic calculations.

We can see that this gives the same result as the the dynamical renormalization group argument

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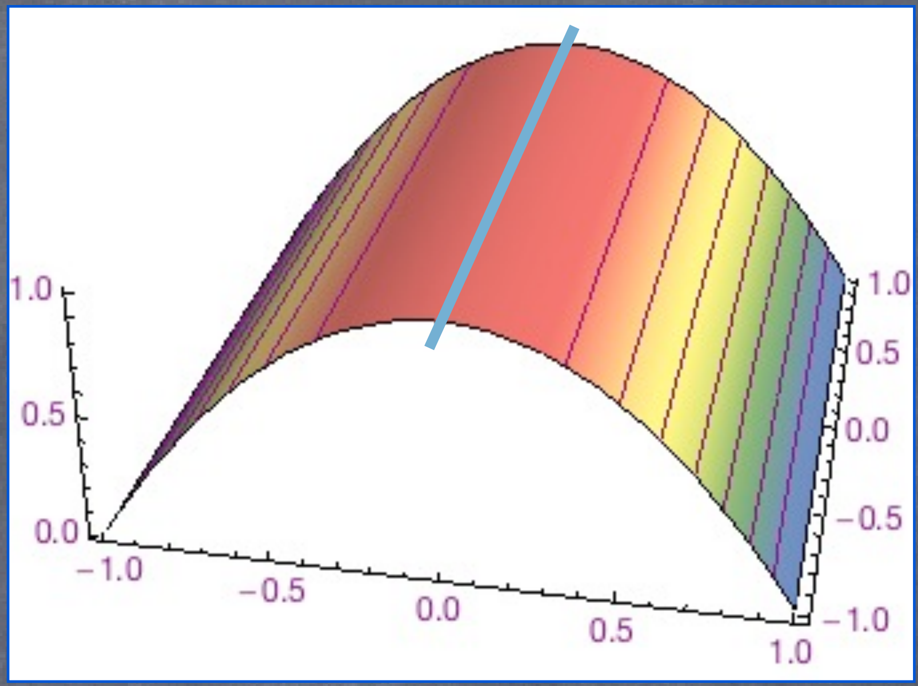
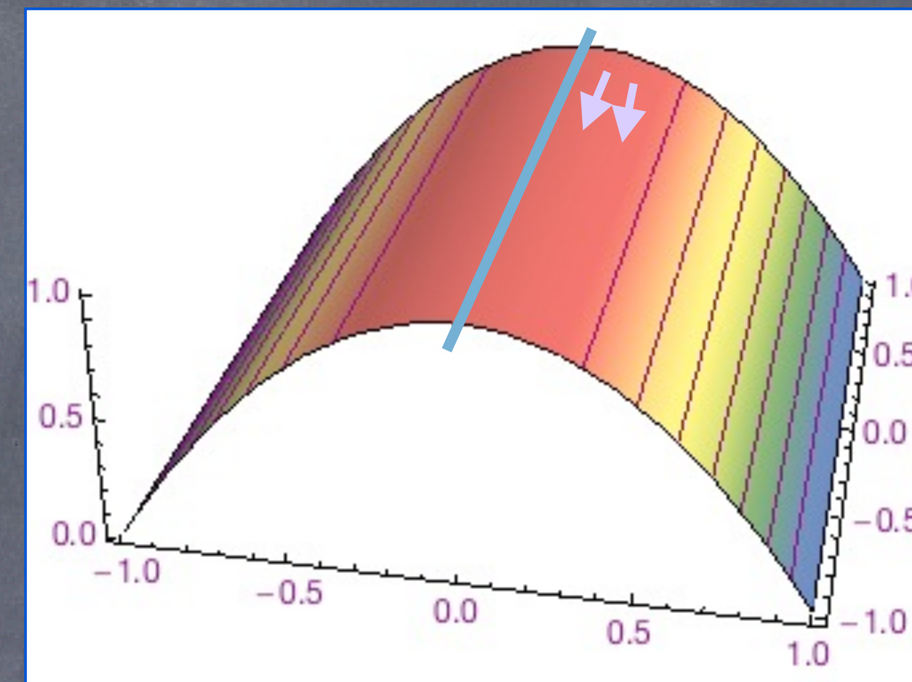


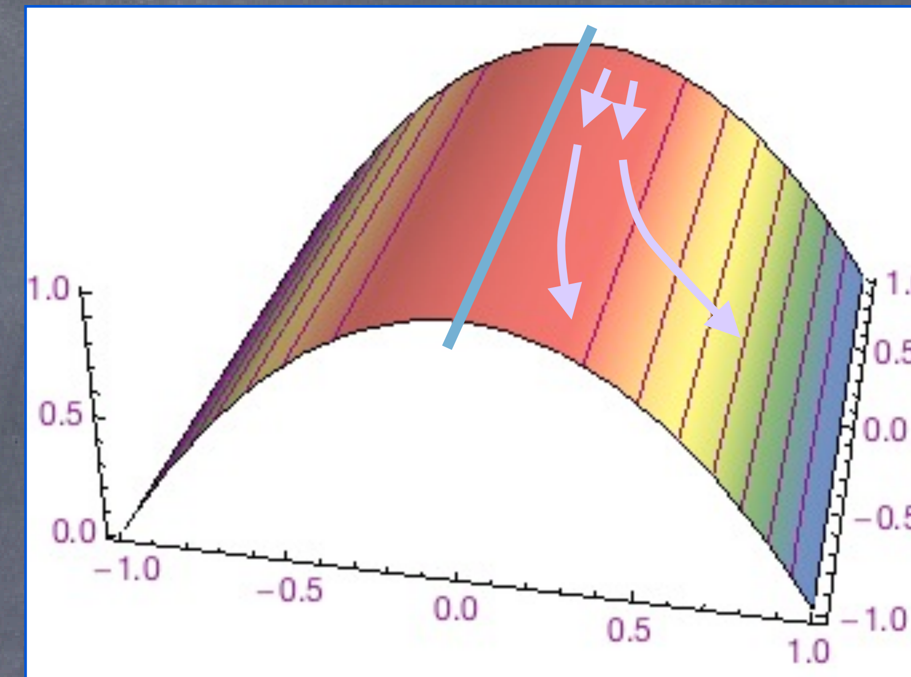
Ridge



Initially the trajectories keep close to each other

Ridge

Eventually they disperse nonlinearly away from the ridge



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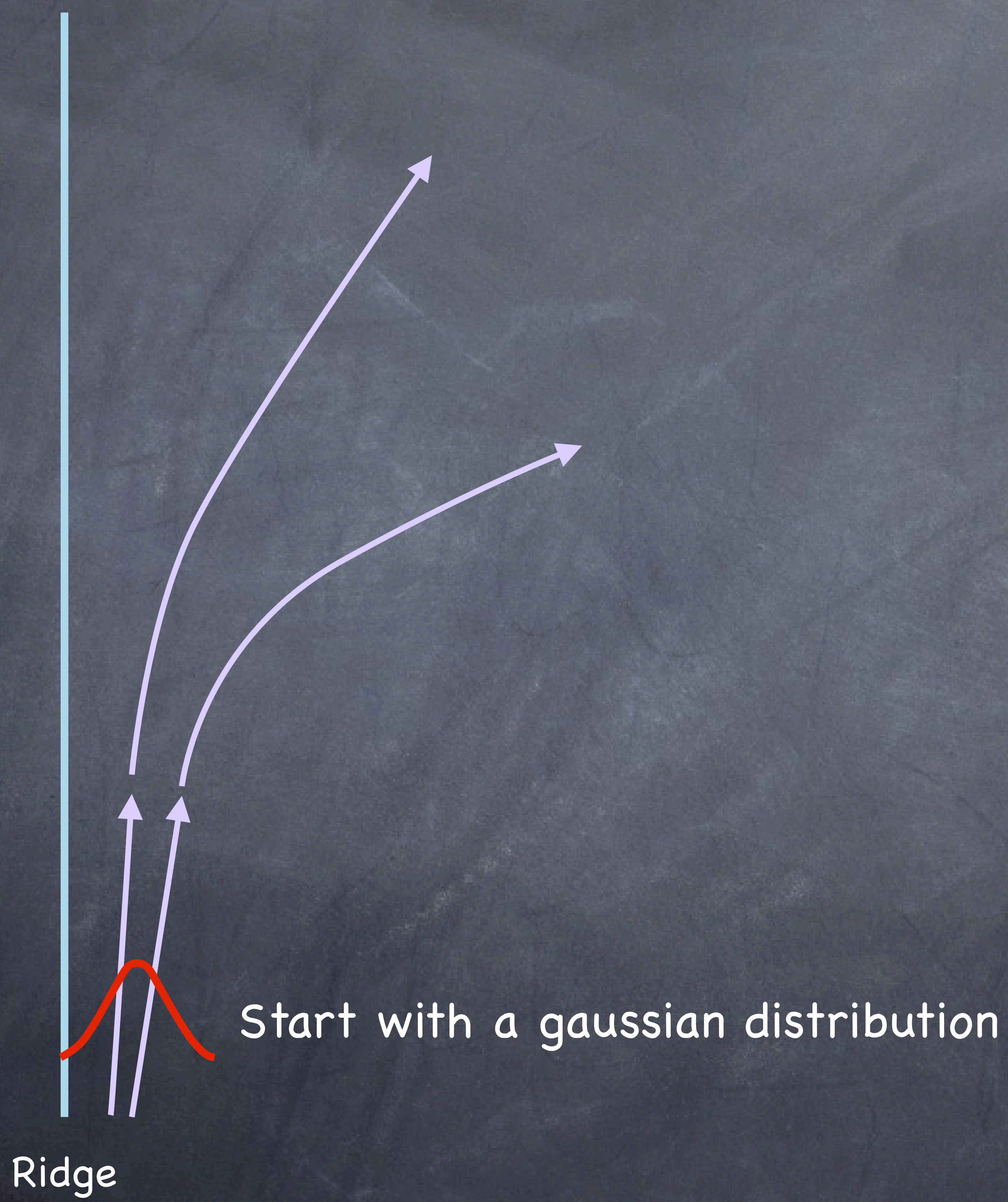
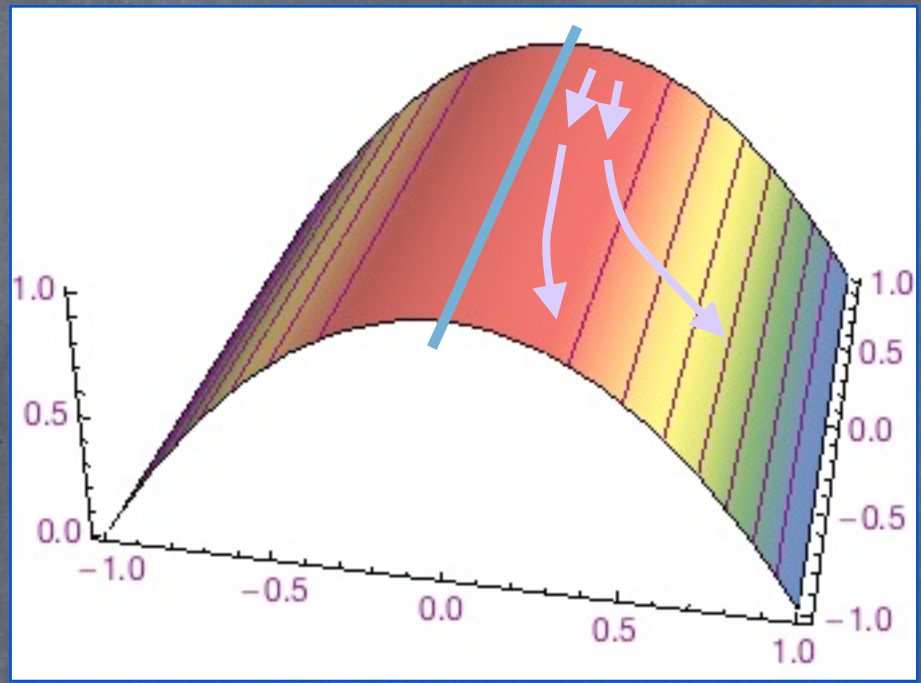
Ridge

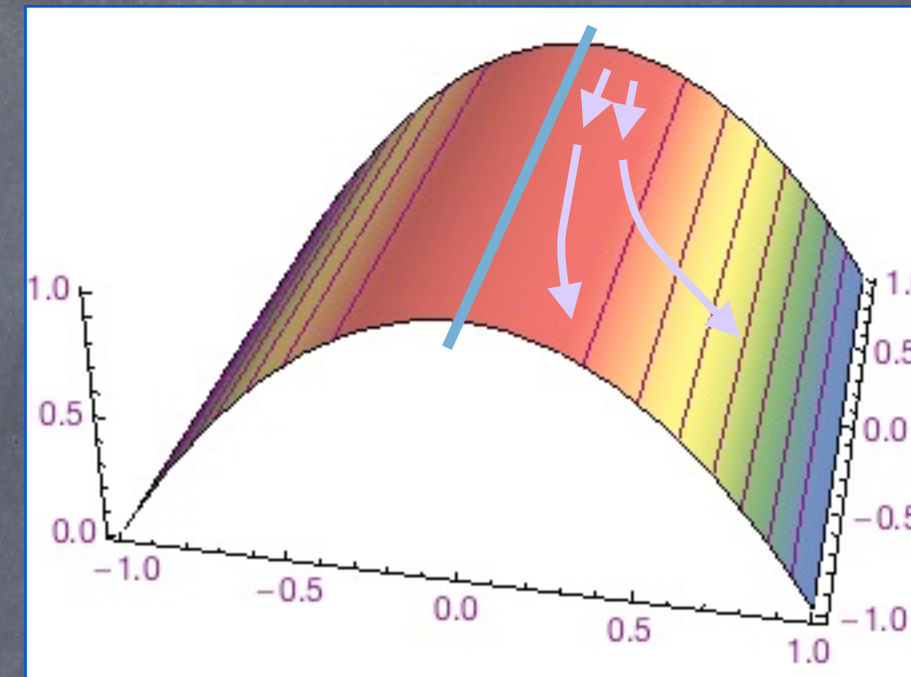

Start with a gaussian distribution
Ridge

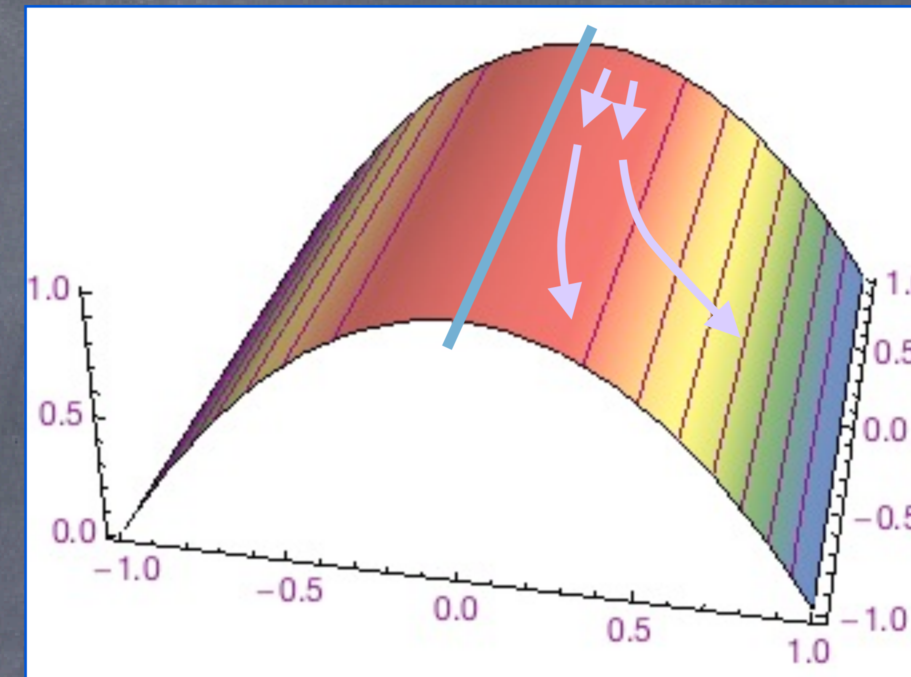

1.0
0.5
0.0
−1.0
−0.5
0.0
0.5
1.0
1.0
0.5
0.0
−0.5
−1.0

Eventually a few trajectories slide away down the hillside, generating a **heavy tail**

The gaussian distribution is preserved in the early phases

Start with a gaussian distribution





Jacobi field $\Gamma_{\alpha i}$

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The gaussian distribution is preserved in the early phases

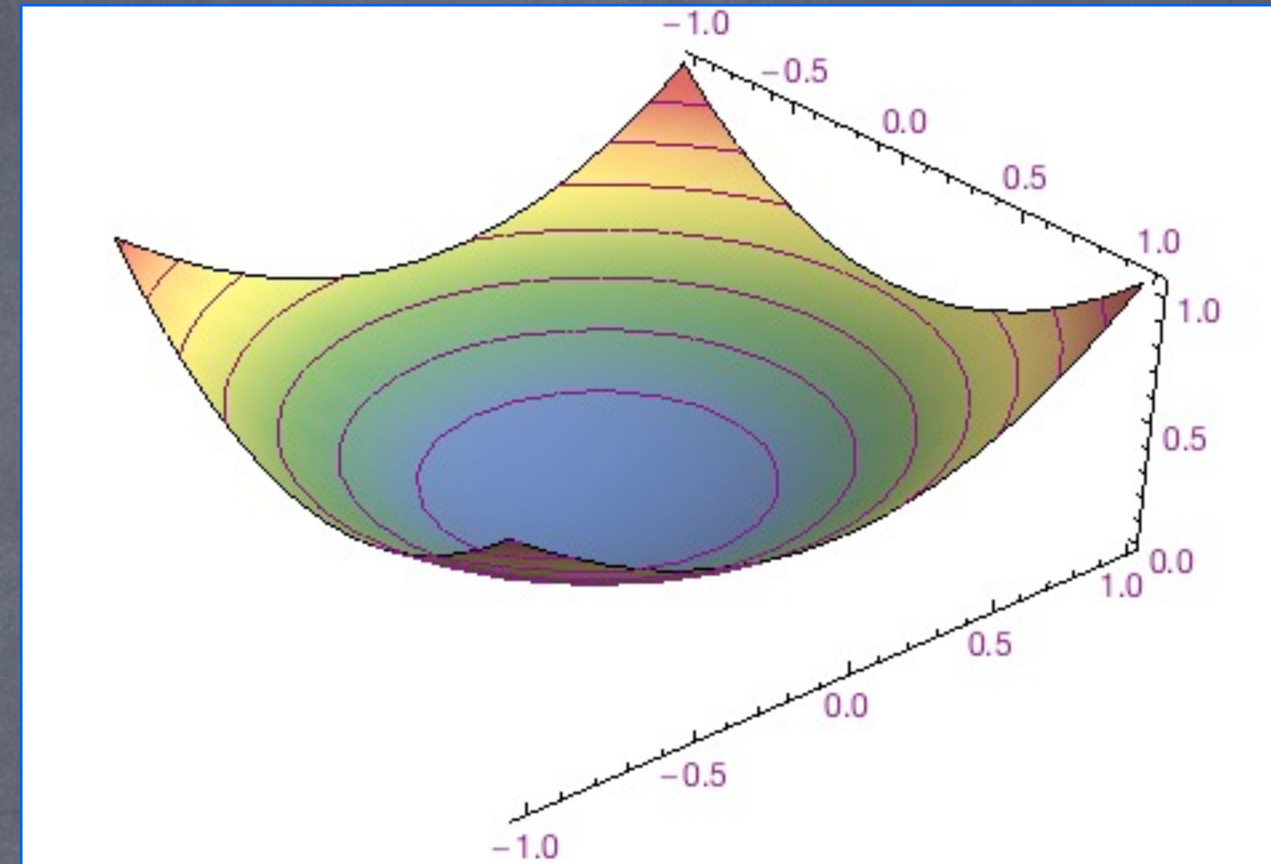
Start with a gaussian distribution

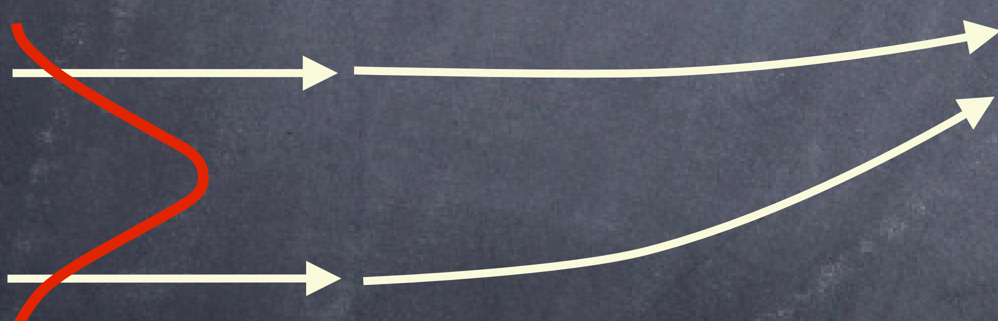
Ridge

(originally García-Bellido & Wands, 1996)

Something similar happens when converging into a valley

$$V = \frac{1}{2}m_\phi^2\phi^2 + g_0\chi + \frac{1}{2}m_\chi^2\chi^2$$





This time, the "uphill" edge of the bundle is compressed towards the centre, which again generates a heavy tail on the "downhill" side.

χ

↑ → ϕ

Direction of valley floor

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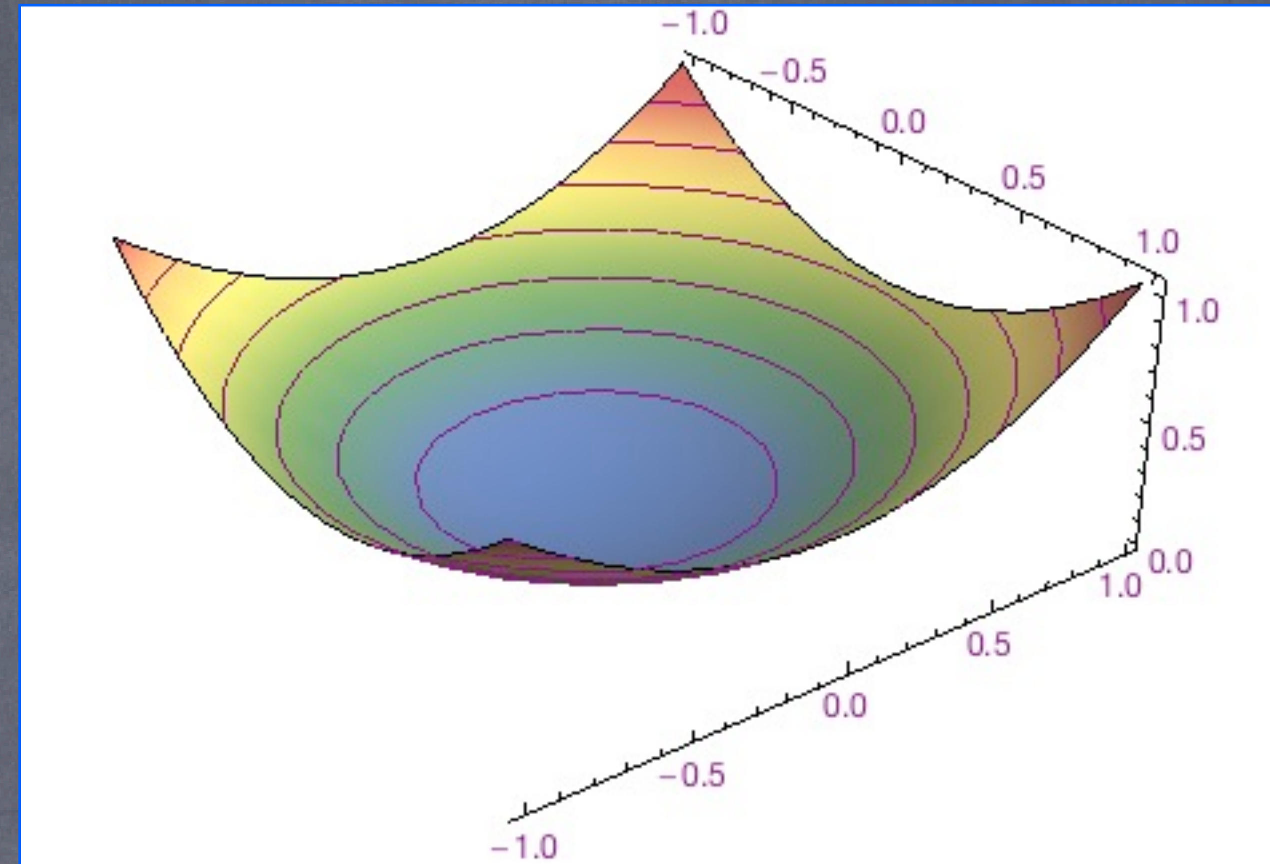

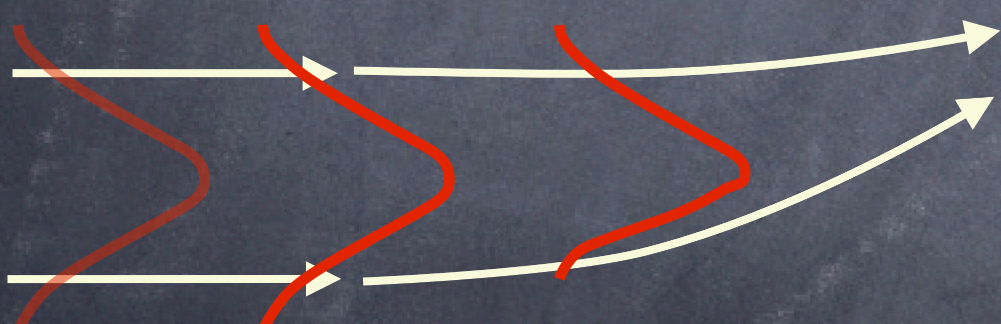

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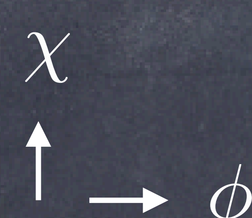
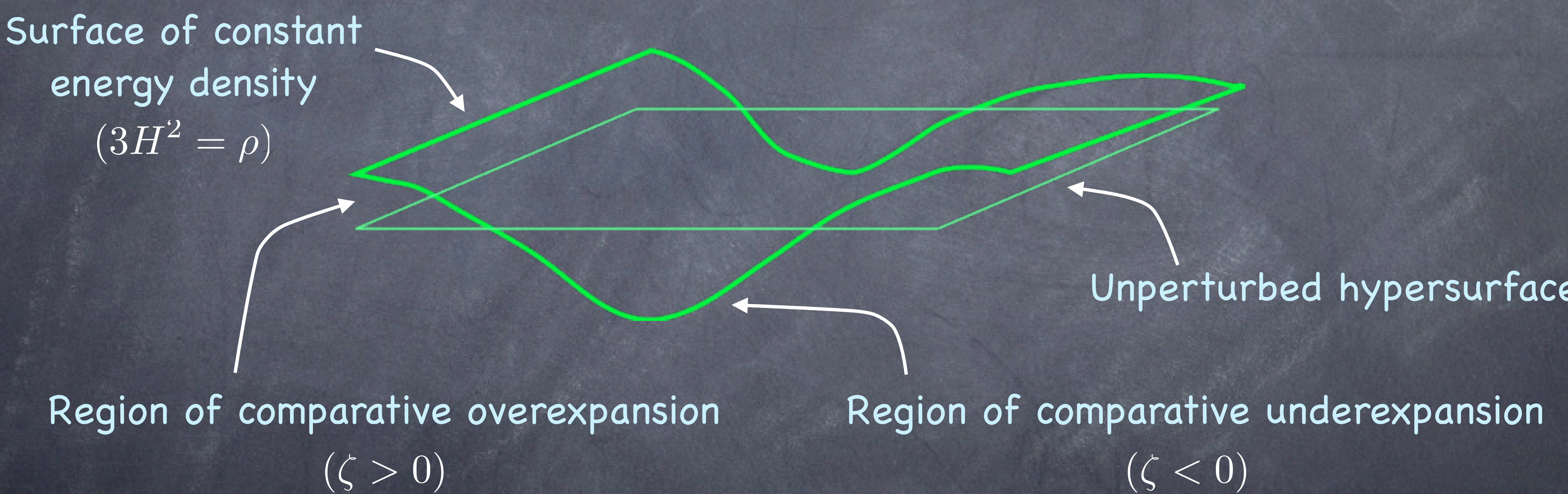


Direction of valley floor

The conclusion is that, to detect light modes, we should look at departures from Gaussian statistics

But in which observable?

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \mathrm{e}^{2\zeta} \mathrm{d}\boldsymbol{x}^2$$



$$a(t) \equiv \exp \int^t H(t')\,\mathrm{d}t' = \exp N(t) \qquad \Rightarrow \qquad a(t)\mathrm{e}^{\zeta} \equiv \exp\{N(t) + \delta N(t)\}$$