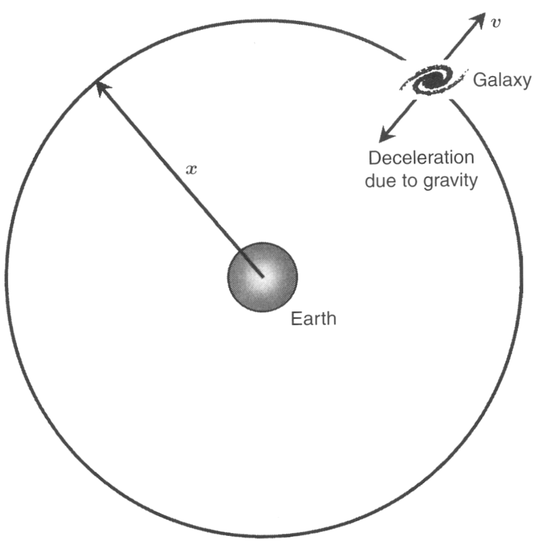
Fundamentals of Cosmology

(2) The Standard Cosmological Models

- Friedman's equations
- Models with finite cosmological constant
- Space-time diagrams revisited

Newtonian Cosmological Models



In 1934, Milne and McCrea showed that the structure of the Friedman equations can be derived using non-relativistic Newtonian dynamics. Consider a galaxy at distance x from the Earth and determine its deceleration due to the gravitational attraction of the matter inside the sphere of radius x centred on the Earth. By Gauss's theorem, because of the spherical symmetry of the distribution of matter within x, we can replace that mass, $M = (4\pi/3)\varrho x^3$, by a point mass at the centre of the sphere and so the deceleration of the galaxy is

$$m\ddot{x} = -\frac{GMm}{x^2} = -\frac{4\pi x \varrho m}{3} \,. \tag{1}$$

The mass of the galaxy m cancels out on either side of the equation, showing that the deceleration refers to the sphere of matter as a whole rather than to any particular galaxy.

We now introduce comoving coordinates. We are dealing with isotropic Universes which expand uniformly. We therefore introduce the concept of comoving distance. If the distance between two points expanding with the Universe is x and r is their separation at the present epoch, we can write $x = (a/a_0)r$ and so take out the expansion of the Universe. I will normally set the scale factor equal to unity at the present epoch, $a_0 = 1$ for simplicity. a is the scale factor.

We can also express the density in terms of its value at the present epoch, $\varrho = \varrho_0 a^{-3}$. Therefore,

$$\boxed{\ddot{a} = -\frac{4\pi G\varrho_0}{3a^2}} \quad \text{or} \quad \boxed{\ddot{a} = -\frac{4\pi G\varrho a}{3},} \tag{2}$$

Multiplying (2) by $\dot{a}$ and integrating, we find

$$\boxed{\dot{a}^2 = \frac{8\pi G\varrho_0}{3a} + \text{constant}} \quad \text{or} \quad \boxed{\dot{a}^2 = \frac{8\pi G\varrho a^2}{3} + \text{constant.}} \tag{3}$$

This Newtonian calculation shows that we can identify the left-hand side of (3) with the kinetic energy of expansion of the fluid and the first term on the right-hand side with its gravitational potential energy.

Why Does this Argument Work?

The above analysis brings out a number of important features of the Friedman world models.

- Note that, because of the assumption of isotropy, local physics is also global physics. The same physics which defines the local behaviour of matter also defines its behaviour on the largest scales. For example, the curvature of space κ within one cubic metre is exactly the same as that on the scale of the Universe itself.

- Although we might appear to have placed the Earth in a rather special position, an observer located on any galaxy would perform exactly the same calculation to work out the deceleration of any other galaxy relative to the observer's galaxy because of the cosmological principle which asserts that all fundamental observers should observe the same large scale features of the Universe at the same epoch. The Newtonian calculation applies for all observers who move in such a way that the Universe appears isotropic to them which is, by definition, for all fundamental observers.

Einstein's Field Equations

In the full GR analysis, Einstein's field equations reduce to the following pair of independent equations.

$$\ddot{a} = -\frac{4\pi G}{3} a \left(\varrho + \frac{3p}{c^2}\right) + \left[\tfrac{1}{3}\Lambda a\right] \; ; \tag{4}$$

$$\dot{a}^2 = \frac{8\pi G \varrho}{3} a^2 - \frac{c^2}{\Re^2} + \left[\tfrac{1}{3}\Lambda a^2\right] \; . \tag{5}$$

In these equations, a is the scale factor, ϱ is the total inertial mass density of the matter and radiation content of the Universe and p the associated total pressure. $\Re$ is the radius of curvature of the geometry of the world model at the present epoch and so the term $-c^2/\Re^2$ is simply a constant of integration. The *cosmological constant* Λ, which has been included in the terms in square brackets in (4) and (5), has had a chequered history since it was introduced by Einstein in 1917.

The Meaning of the Term $\varrho + \frac{3p}{c^2}$

Let us look more closely at the meanings of the various terms. Equation (5) is referred to as *Friedman's equation* and has the form of an energy equation, the term on the left-hand side corresponding to the kinetic energy of the expanding fluid and the first term on the right-hand side to its gravitational potential energy. The First Law of Thermodynamics in its relativistic form needs to be built into this equation. We can write it in the usual form

$$\mathrm{d}U = -p\,\mathrm{d}V\ . \tag{6}$$

We need to formulate the first law in such a way that it is applicable for relativistic and non-relativistic fluids and so we write the internal energy U as the sum of all the terms which can contribute to the total energy of the fluid in the relativistic sense. Thus, the total internal energy consists of the fluid's rest mass energy, its kinetic energy, its thermal energy and so on. If we write the sum of these energies as $\varepsilon_{\mathrm{tot}} = \sum_i \varepsilon_i$, the internal energy is $\varepsilon_{\mathrm{tot}} V$ and so, differentiating (6) with respect to a, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}a}(\varepsilon_{\mathrm{tot}} V) = -p\,\frac{\mathrm{d}V}{\mathrm{d}a}\ . \tag{7}$$

Now, $V \propto a^3$ and so, differentiating, we find

$$\frac{\mathrm{d}\varepsilon_{\mathrm{tot}}}{\mathrm{d}a} + 3\frac{(\varepsilon_{\mathrm{tot}} + p)}{a} = 0 \; . \tag{8}$$

This result can be expressed in terms of the inertial mass density associated with the total energy $\varepsilon_{\mathrm{tot}} = \varrho c^2$ and so (8) can also be written

$$\frac{\mathrm{d}\varrho}{\mathrm{d}a} + 3\frac{\left(\varrho + \frac{p}{c^2}\right)}{a} = 0 \; . \tag{9}$$

This is the type of density ϱ which should be included in (4) and (5).

These equations lead to a number of important results which we will use repeatedly in what follows. First of all, suppose the fluid is very 'cold' in the sense that $p \ll \varrho_0 c^2$, where ϱ_0 is its rest mass density. Then, setting $p = 0$ and $\varepsilon_0 = Nmc^2$, where N is the number density of particles of rest mass m, we find

$$\frac{\mathrm{d}N}{\mathrm{d}a} + \frac{3N}{a} = 0 \quad \text{and so} \quad N = N_0 a^{-3} \; , \tag{10}$$

that is, the continuity equation.

The Thermal Gas

Next, the thermal pressure of non-relativistic matter can be included into (8). For essentially all our purposes, we will be dealing with monatomic gases or plasmas for which the thermal energy is $\varepsilon_{\mathrm{th}} = \frac{3}{2}NkT$ and $p = NkT$.

Then, substituting $\varepsilon_{\mathrm{tot}} = \frac{3}{2}NkT + Nmc^2$ and $p = NkT$ into (8), we find

$$\frac{\mathrm{d}}{\mathrm{d}a}\left(\tfrac{3}{2}NkT + Nmc^2\right) + 3\left(\frac{\frac{5}{2}NkT + Nmc^2}{a}\right) = 0\,,$$

$$\frac{\mathrm{d}(NkT)}{\mathrm{d}a} + \frac{5NkT}{a} = 0 \quad \text{and so} \quad NkT = N_0 k T_0 a^{-5}\,. \qquad (11)$$

Since $N = N_0 a^{-3}$, we find the standard result for the adiabatic expansion of a monatomic gas with ratio of specific heats $\gamma = 5/3$, $T \propto a^{-2}$. More generally, if the ratio of specific heats of the gas is γ, the temperature changes as

$$T \propto a^{-3(\gamma-1)}.$$

Peculiar Velocities and the Relativistic Gas

We can deduce another important result from $T \propto a^{-2}$. If we write $\varepsilon_{\mathrm{th}} = \frac{1}{2}Nm\langle v^2\rangle$, we find that

$$\langle v^2\rangle \propto a^{-2}.$$

Thus, the random velocities of the particles of the gas decrease as $v \propto a^{-1}$. This result applies equally to the random motions of galaxies relative to the mean Hubble flow, what are known as the *peculiar velocities* of galaxies, v_{pec}. Therefore, as the Universe expands, we expect the peculiar velocities of galaxies to decrease as $v_{\mathrm{pec}} \propto a^{-1}$.

Finally, in the case of a gas of *ultrarelativistic particles*, or a *gas of photons*, we can write $p = \frac{1}{3}\varepsilon_{\mathrm{tot}}$. Therefore,

$$\frac{\mathrm{d}\varepsilon_{\mathrm{tot}}}{\mathrm{d}a} + \frac{4\varepsilon_{\mathrm{tot}}}{a} = 0 \quad \text{and so} \quad \varepsilon_{\mathrm{tot}} \propto a^{-4}\,. \qquad (12)$$

In the case of a gas of photons, $\varepsilon_{\mathrm{rad}} = \sum Nh\nu$ and, since $N \propto a^{-3}$, we find $\nu \propto a^{-1}$. The purpose of these calculations is to show how (8) and (9) correctly describe the law of conservation of energy for both relativistic and non-relativistic gases.

Let us now return to the analysis of (5). Differentiating

$$\dot{a}^2 = \frac{8\pi G \varrho}{3} a^2 - \frac{c^2}{\Re^2} + \left[\tfrac{1}{3}\Lambda a^2\right] \ . \tag{13}$$

with respect to time and dividing through by $\dot{a}$, we find

$$\ddot{a} = \frac{4\pi G a^2}{3} \frac{\mathrm{d}\varrho}{\mathrm{d}a} + \frac{8\pi G \varrho a^2}{3} + \left[\tfrac{1}{3}\Lambda a\right] \ . \tag{14}$$

Now, substituting the expression for $\mathrm{d}\varrho/\mathrm{d}a$ from (9), we find

$$\boxed{\ddot{a} = -\frac{4\pi G}{3} a \left(\varrho + \frac{3p}{c^2}\right) + \left[\tfrac{1}{3}\Lambda a\right] \ ,} \tag{15}$$

that is, we recover (4).

Thus, equation (15) has the form of a force equation, but, as we have shown, it also incorporates the relativistic form of the First Law of Thermodynamics as well. This pressure term can be considered a ‘relativistic correction’ to the inertial mass density, but it is unlike normal pressure forces which depend upon the gradient of the pressure and, for example, hold up the stars. The term $\varrho + (3p/c^2)$ can be thought of as playing the role of an *active gravitational mass density*.

The Cosmological Constant Λ

Einstein's introduction of the cosmological constant predated Hubble's discovery of the expansion of the distribution of galaxies. In 1917, Einstein introduced the Λ-term in order to incorporate *Mach's principle* into General Relativity - namely that the local inertial frame of reference should be defined relative to the distant stars. In the process, he derived the first fully self-consistent cosmological model - the static Einstein model of the Universe.

Equation (4) is

$$\ddot{a} = -\frac{4\pi G}{3} a \left(\varrho + \frac{3p}{c^2}\right) + \left[\tfrac{1}{3}\Lambda a\right] \,. \qquad (16)$$

Einstein's model is static and so $\ddot{a} = 0$ and the model is a 'dust model' in which the pressure is taken to be zero. Therefore,

$$\frac{4\pi G}{3} a\varrho = \tfrac{1}{3}\Lambda a \quad \text{or} \quad \boxed{\Lambda = 4\pi G\varrho} \,. \qquad (17)$$

Einstein's perspective was that this formula shows that there would be no solutions of his field equations unless the cosmological constant was finite. If Λ were zero, the Universe would be empty.

The Cosmological Constant Λ

Let us consider the first of the field equations with finite Λ.

$$\ddot{a} = -\frac{4\pi G}{3} a \left(\varrho + \frac{3p}{c^2} \right) + \tfrac{1}{3} \Lambda a \; . \tag{18}$$

Inspection of (18) gives insight into the physical meaning of the cosmological constant. Even in an empty universe, with $\varrho = 0, p = 0$, there is a net force acting on a test particle. If Λ is positive, the term may be thought of as the ‘repulsive force of a vacuum’, the repulsion being relative to an absolute geometrical frame of reference. There is no obvious interpretation of this term in term of classical physics. There is, however, a natural interpretation in the context of quantum field theory.

A key development has been the introduction of *Higgs* fields into the theory of weak interactions. These were introduced in order to eliminate singularities in the theory and to endow the $W^{\pm}$ and Z^0 bosons with masses. Precise measurement of the masses of these particles at CERN has confirmed the theory very precisely and the recent announcement of the discovery of the Higgs boson was a real triumph. The Higgs fields are *scalar fields*, unlike the vector fields of electromagnetism or the tensor fields of General Relativity. They have negative pressure equations of state $p = -\varrho c^2$.

The Cosmological Constant Λ

In the modern picture of the vacuum, there are zero-point fluctuations associated with the zero point energies of all quantum fields. The stress–energy tensor of a vacuum has a negative pressure equation of state, $p = -\varrho c^2$. This pressure may be thought of as a 'tension' rather than a pressure. When such a vacuum expands, the work done $p\,\mathrm{d}V$ in expanding from V to $V + \mathrm{d}V$ is just $-\varrho c^2\,\mathrm{d}V$ so that, during the expansion, the mass-energy density of the negative pressure field remains constant.

We can find the same result from (9).

$$\frac{\mathrm{d}\varrho}{\mathrm{d}a} + 3\frac{\left(\varrho + \frac{p}{c^2}\right)}{a} = 0\,.$$

It can be seen that, if the vacuum energy density is to remain constant, it follows that $p = -\varrho c^2$.

We can now relate ϱ_{V} to the value of Λ. We can now set $\Lambda = 0$ and instead include the energy and pressure of the vacuum fields into equation (18).

The Cosmological Constant Λ

$$\ddot{a} = -\frac{4\pi G a}{3}\left(\varrho_{\rm m} + \varrho_{\rm v} + \frac{3p_{\rm v}}{c^2}\right) , \qquad (19)$$

where, in place of the Λ-term, we have included the density of ordinary mass $\varrho_{\rm m}$ and the mass density $\varrho_{\rm v}$ and pressure $p_{\rm v}$ of the vacuum fields. Since $p_{\rm v} = -\varrho_{\rm v} c^2$, it follows that

$$\ddot{a} = -\frac{4\pi G a}{3}(\varrho_{\rm m} - 2\varrho_{\rm v}) . \qquad (20)$$

As the Universe expands, $\varrho_{\rm m} = \varrho_0/a^3$ and $\varrho_{\rm v}$ = constant. Therefore,

$$\ddot{a} = -\frac{4\pi G \varrho_0}{3a^2} + \frac{8\pi G \varrho_{\rm v} a}{3} . \qquad (21)$$

Equation (21) has precisely the same dependence upon a as of the 'cosmological term' and so we can formally identify the cosmological constant with the vacuum mass density.

$$\Lambda = 8\pi G \varrho_{\rm v} . \qquad (22)$$

Density Parameters in the Matter and Vacuum Fields

Therefore, at the present epoch, $a = 1$, the first field equation becomes

$$\ddot{a}(t_0) = -\frac{4\pi G \varrho_0}{3} + \frac{8\pi G \varrho_v}{3} . \qquad (23)$$

It is convenient to express densities in terms of the *critical density* ϱ_c which is defined to be

$$\varrho_c = (3H_0^2/8\pi G) = 1.88 \times 10^{-26}\, h^2 \text{ kg m}^{-3} . \qquad (24)$$

This is the density of the critical Einstein-de Sitter world model. Then, the actual density of the model ϱ_0 at the present epoch can be referred to this value through a *density parameter* $\Omega_0 = \varrho_0/\varrho_c$.

$$\Omega_0 = \frac{8\pi G \varrho_0}{3H_0^2} . \qquad (25)$$

The subscript 0 has been attached to Ω because the critical density ϱ_c changes with cosmic epoch, as does Ω. It is convenient to refer any cosmic density to ϱ_c. For example, we will often refer to the density parameter of baryons, Ω_B, or of visible matter, Ω_{vis}, or of dark matter, Ω_{dark}, and so on – these are convenient ways of describing the relative importance of different contributions to Ω_0.

Density Parameter in the Matter

The dynamical equations (4) and (5) with $\Lambda = 0$ therefore become

$$\ddot{a} = -\frac{\Omega_0 H_0^2}{2a^2} \; ; \tag{26}$$

$$\dot{a}^2 = \frac{\Omega_0 H_0^2}{a} - \frac{c^2}{\Re^2} \, . \tag{27}$$

Several important results can be deduced from these equations. If we set the quantities in (27) equal to their values at the present epoch, $t = t_0$, $a = 1$ and $\dot{a} = H_0$, we find

$$\Re = \frac{c/H_0}{(\Omega_0 - 1)^{1/2}} \qquad \text{and} \qquad \kappa = \frac{(\Omega_0 - 1)}{(c/H_0)^2}. \tag{28}$$

This last result shows that there is a one-to-one relation between the density of the Universe and its spatial curvature, $\Re$, one of the most beautiful results of the Friedman world models with $\Lambda = 0$.

The Dynamics of the Models with $\Lambda = 0$

To understand the solutions of (27), we substitute (28) into (27) to find the following expression for $\dot{a}$

$$\dot{a}^2 = H_0^2 \left[\Omega_0 \left(\frac{1}{a} - 1 \right) + 1 \right] . \tag{29}$$

In the limit of large values of a, $\dot{a}^2$ tends to $\dot{a}^2 = H_0^2(1 - \Omega_0)$.

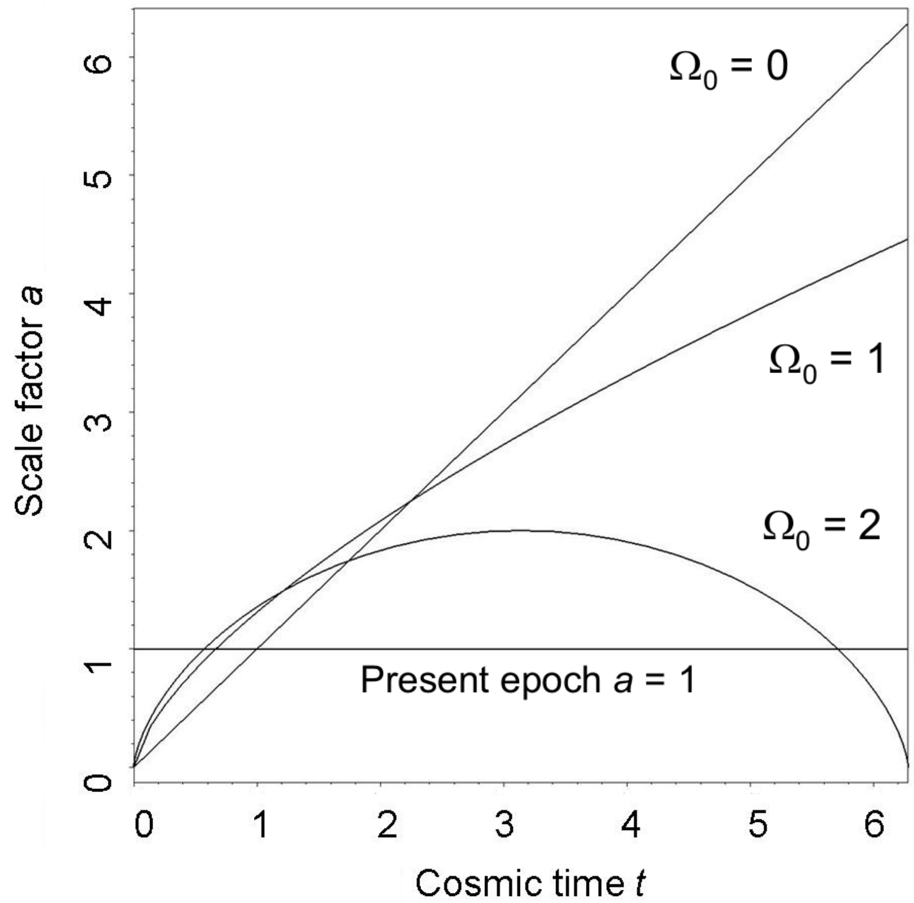
- The models having $\Omega_0 < 1$ have open, hyperbolic geometries and expand to $a = \infty$. They continue to expand with a finite velocity at $a = \infty$ with $\dot{a} = H_0(1 - \Omega_0)^{1/2}$;

- The models with $\Omega_0 > 1$ have closed, spherical geometry and stop expanding at some finite value of $a = a_{\text{max}}$ – they have 'imaginary expansion rates' at infinity. They reach the maximum value of the scale factor after a time

$$t_{\text{max}} = \frac{\pi \Omega_0}{2 H_0 (\Omega_0 - 1)^{3/2}} . \tag{30}$$

  These models collapse to an infinite density after a finite time $t = 2\, t_{\text{max}}$, an event sometimes referred to as the 'big crunch';

The Dynamics of the Models with $\Lambda = 0$



- The model with $\Omega_0 = 1$ separates the open from the closed models and the collapsing models from those which expand forever. This model is often referred to as the *Einstein–de Sitter* or the *critical model*. The velocity of expansion tends to zero as a tends to infinity. It has a particularly simple variation of $a(t)$ with cosmic epoch,

$$a = \left(\frac{t}{t_0}\right)^{2/3} \qquad \kappa = 0, \qquad (31)$$

where the present age of the world model is $t_0 = (2/3)H_0^{-1}$.

Density Parameter in the Vacuum Fields

A density parameter associated with $\varrho_{\rm V}$ can now be introduced, in exactly the same way as the density parameter Ω_0 was defined.

$$\Omega_\Lambda = \frac{8\pi G \varrho_{\rm V}}{3H_0^2} \quad \text{and so} \quad \Lambda = 3H_0^2 \Omega_\Lambda \, . \tag{32}$$

The dynamical equations (4) and (5) can now be written

$$\ddot{a} = -\frac{\Omega_0 H_0^2}{2a^2} + \Omega_\Lambda H_0^2 a \, ; \tag{33}$$

$$\dot{a}^2 = \frac{\Omega_0 H_0^2}{a} - \frac{c^2}{\Re^2} + \Omega_\Lambda H_0^2 a^2 \, . \tag{34}$$

A traditional way of rewriting these relations is in terms of a *deceleration parameter* q_0 defined by $q_0 = -\ddot{a}/\dot{a}^2$ at the present epoch. Then, in terms of Ω_0 and Ω_Λ, we find from (33),

$$\boxed{q_0 = \frac{\Omega_0}{2} - \Omega_\Lambda \, .} \tag{35}$$

Density Parameters in Matter and Vacuum Fields

We can now substitute the values of a and $\dot{a}$ at the present epoch, $a = 1$ and $\dot{a} = H_0$, into (34) to find the relation between the curvature of space, Ω_0 and Ω_Λ.

$$\frac{c^2}{\Re^2} = H_0^2[(\Omega_0 + \Omega_\Lambda) - 1] \; , \tag{36}$$

or

$$\kappa = \frac{1}{\Re^2} = \frac{[(\Omega_0 + \Omega_\Lambda) - 1]}{(c^2/H_0^2)} \; . \tag{37}$$

A common practice is to introduce a density parameter associated with the curvature of space at the present epoch Ω_K such that

$$\Omega_K = -\frac{c^2}{H_0^2 \Re^2} \tag{38}$$

Then, equation (37) becomes

$$\Omega_0 + \Omega_\Lambda + \Omega_K = 1 \; . \tag{39}$$

Density Parameters in Matter and Vacuum Fields

Thus, the condition that the spatial sections are flat Euclidean space becomes

$$(\Omega_0 + \Omega_\Lambda) = 1 \ . \qquad (40)$$

The radius of curvature $R_{\rm c}$ of the spatial sections of these models change with scale factor as $R_{\rm c} = a\Re$ and so, if the space curvature is zero now, it must have been zero at all times in the past. This is one of the great attractions of the simplest inflationary picture of the early Universe.

The Dynamics of World Models with $\Lambda \neq 0$

The dynamics of world models with $\Lambda \neq 0$ are of special importance in the light of the most recent estimates of the values of cosmological parameters. First of all, we discuss some general considerations of the dynamics of these models and then look in more detail at the range of models which are likely to be relevant for our future studies.

Models with $\Lambda < 0$ are not of a great deal of interest because the net effect is to incorporate an attractive force in addition to gravity which slows down the expansion of the Universe. The one difference from the models with $\Lambda = 0$ is that, no matter how small the values of Ω_Λ and Ω_0 are, the universal expansion is eventually reversed, as may be seen by inspection of (21) if ρ_v is taken to be negative.

The Dynamics of World Models with $\Lambda \neq 0$

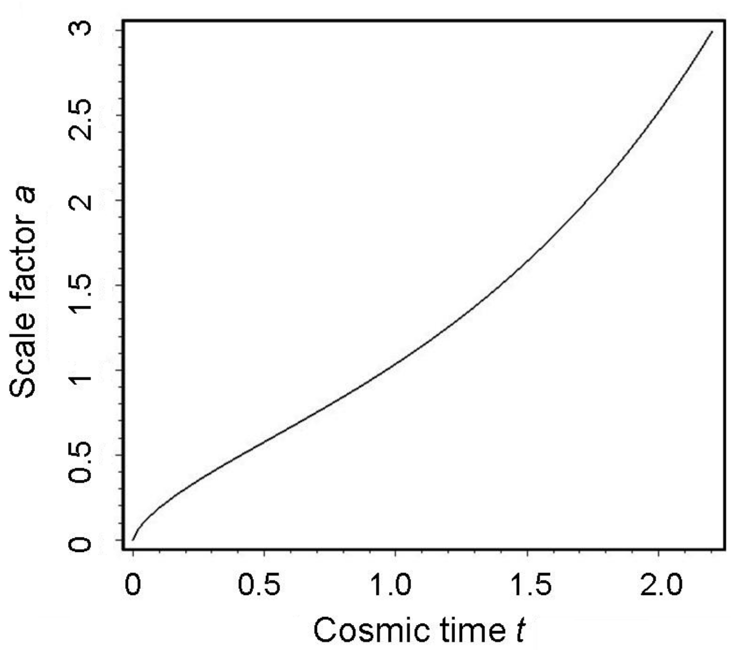
Models with $\Lambda > 0$, $\Omega_\Lambda > 0$ result in a repulsive force which opposes the attractive force of gravity. There is a minimum rate of expansion $\dot{a}_{\rm min}$ at scale factor and minimum rate of expansion:

$$a_{\rm min} = (\Omega_0/2\Omega_\Lambda)^{1/3}\,, \qquad (41)$$

$$\dot{a}^2_{\rm min} = \frac{3H_0^2}{2}(2\Omega_\Lambda\Omega_0^2)^{1/3} - \frac{c^2}{\Re^2}\,. \qquad (42)$$

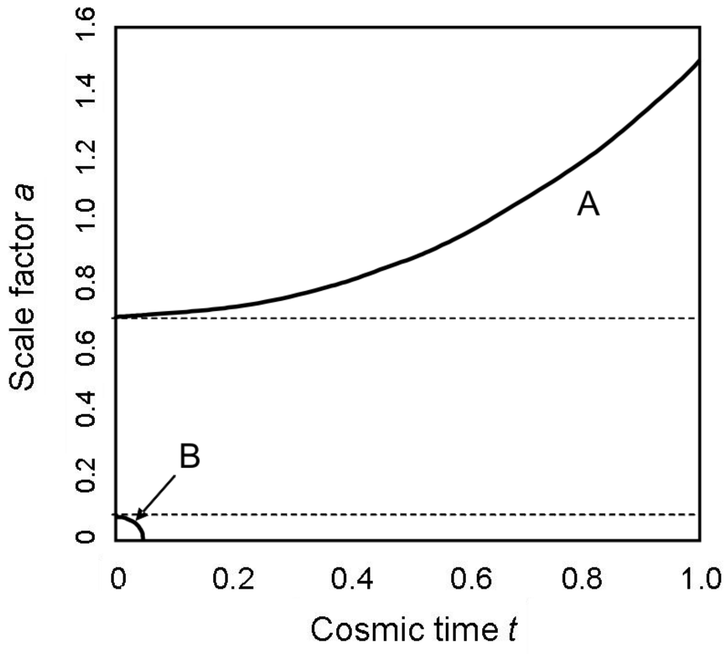
If the right-hand side of (42) is greater than zero, the dynamical behaviour shown in the diagram is found. For large values of a, the dynamics become those of the de Sitter universe

$$a(t) \propto \exp\left[\left(\frac{\Lambda}{3}\right)^{1/2} t\right] = \exp\left(\Omega_\Lambda^{1/2} H_0 t\right)\,. \qquad (43)$$



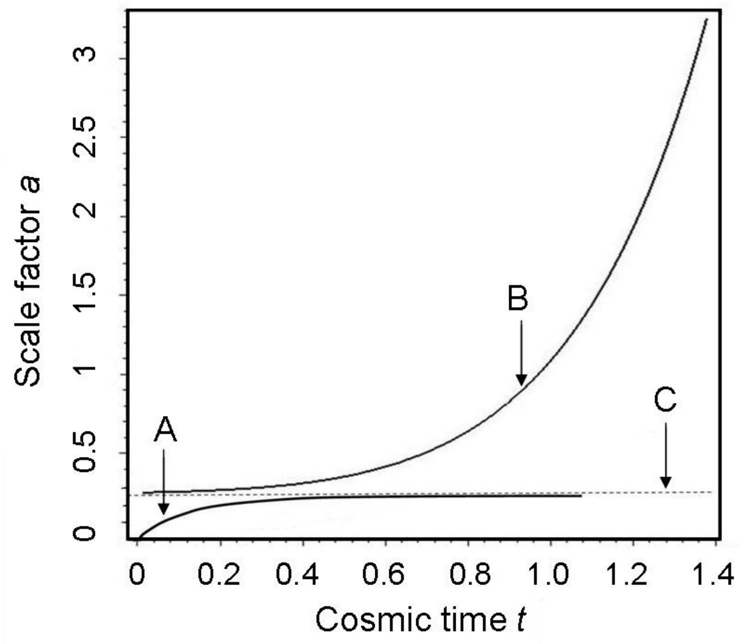
The Dynamics of World Models with $\Lambda \neq 0$

If the right-hand side of (42) is less than zero, there exists a range of scale factors for which no solution exists and it can be shown readily that the function $a(t)$ has two branches. For the branch B, the Universe never expanded to sufficiently large values of a that the repulsive effect of the Λ term can prevent the Universe collapsing. In the case of branch A, the dynamics are dominated by the Λ term – the repulsive force is so strong that the Universe never contracted to such a scale that the attractive force of gravity could overcome its influence. In the latter model, there was no initial singularity – the Universe 'bounced' under the influence of the Λ-term.



The Dynamics of World Models with $\Lambda \neq 0$

The most interesting cases are those for which $\dot{a}_{\rm min} \approx 0$. The case $\dot{a}_{\rm min} = 0$ is known as the *Eddington–Lemaître model*. A, the Universe expanded from an origin at some finite time in the past and will eventually attain a stationary state in the infinite future; B, the Universe is expanding away from a stationary solution in the infinite past. The stationary state C is unstable because, if it is perturbed, the Universe moves either onto branch B, or onto the collapsing variant of branch A. In Einstein's static Universe, the stationary phase occurs at the present day. From (42), the value of Λ corresponding to $\dot{a}_{\rm min} = 0$ is



$$\Lambda = \frac{3}{2}\Omega_0 H_0^2 (1+z_{\rm c})^3 \quad \text{or} \quad \Omega_\Lambda = \frac{\Omega_0}{2}(1+z_{\rm c})^3 \,, \tag{44}$$

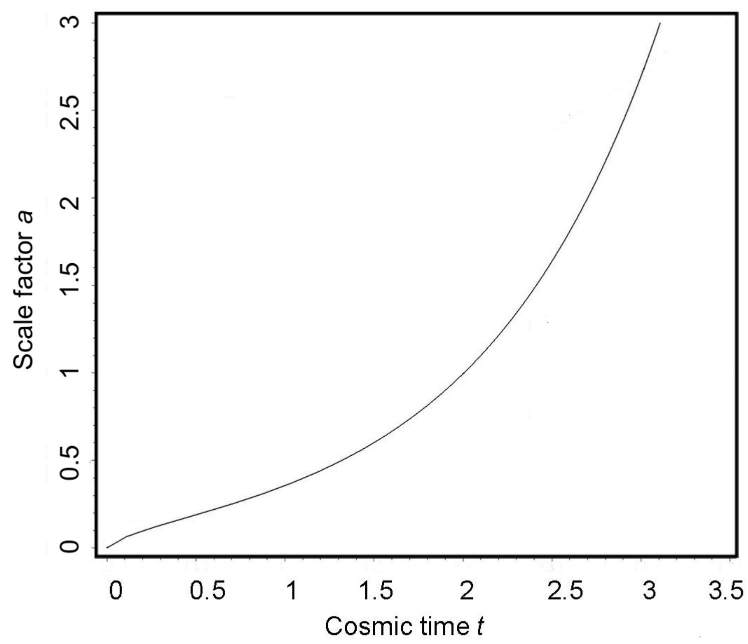
where $z_{\rm c}$ is the redshift of the stationary state.

The Dynamics of World Models with $\Lambda \neq 0$

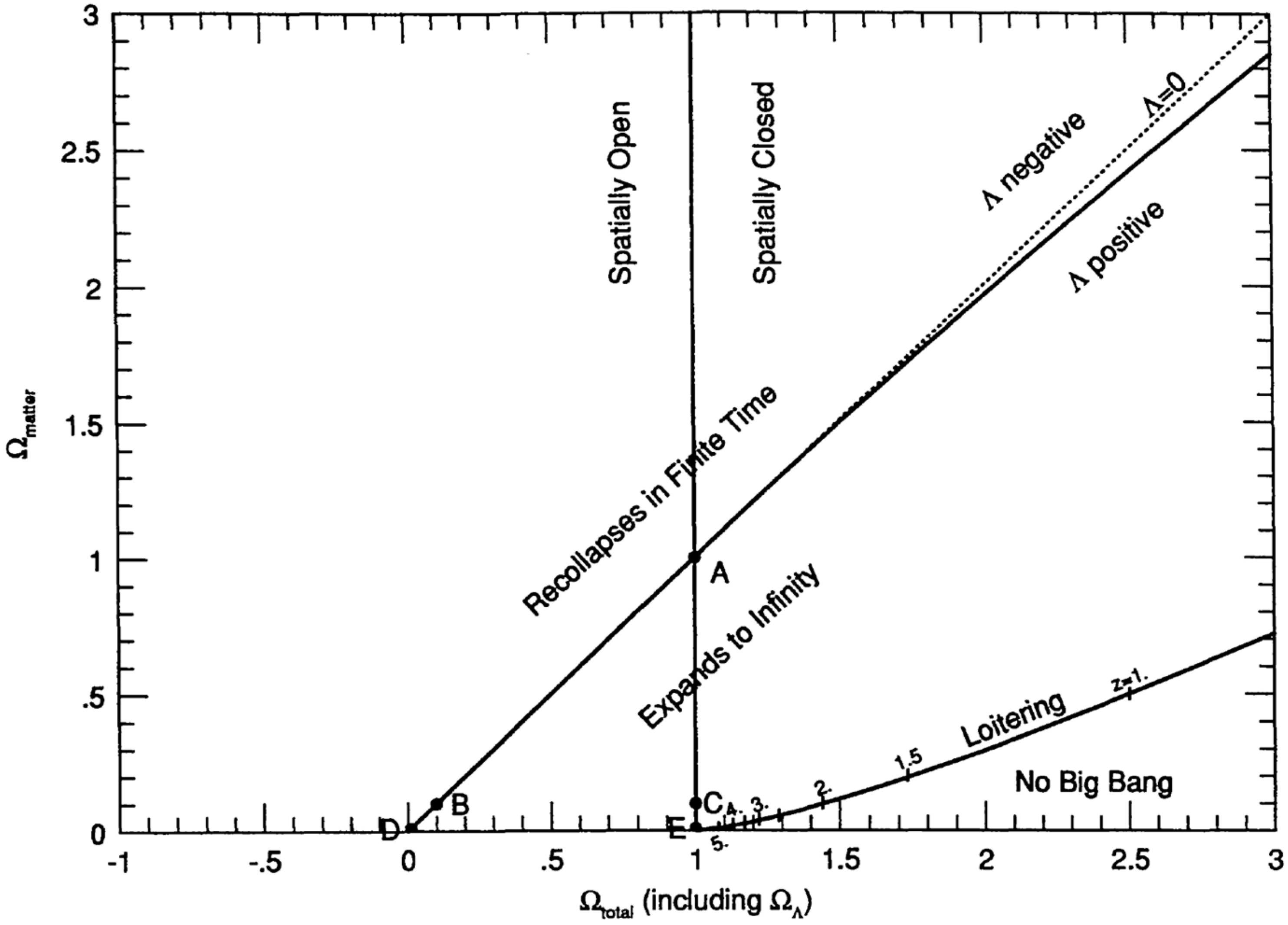
The static Eddington–Lemaître models have $\dot{a} = 0$ for all time and we can find a one-to-one relation between the mean density of matter in the Universe Ω_0 and the redshift of the stationary phase z_c.

$$\Omega_0 = \frac{2}{(1+z_c)^3 - 3(1+z_c) + 2} = \frac{2}{z_c^2(z_c+3)} \, . \tag{45}$$

This calculation is largely of academic interest nowadays. If a stationary, or near-stationary, state had occurred, the fact that galaxies and quasars are now observed with redshifts $z > 6$ suggests that $z_c > 6$ and so $\Omega_0 \leq 0.01$, which is at least an order of magnitude less than the total mass density in dark matter at the present epoch.

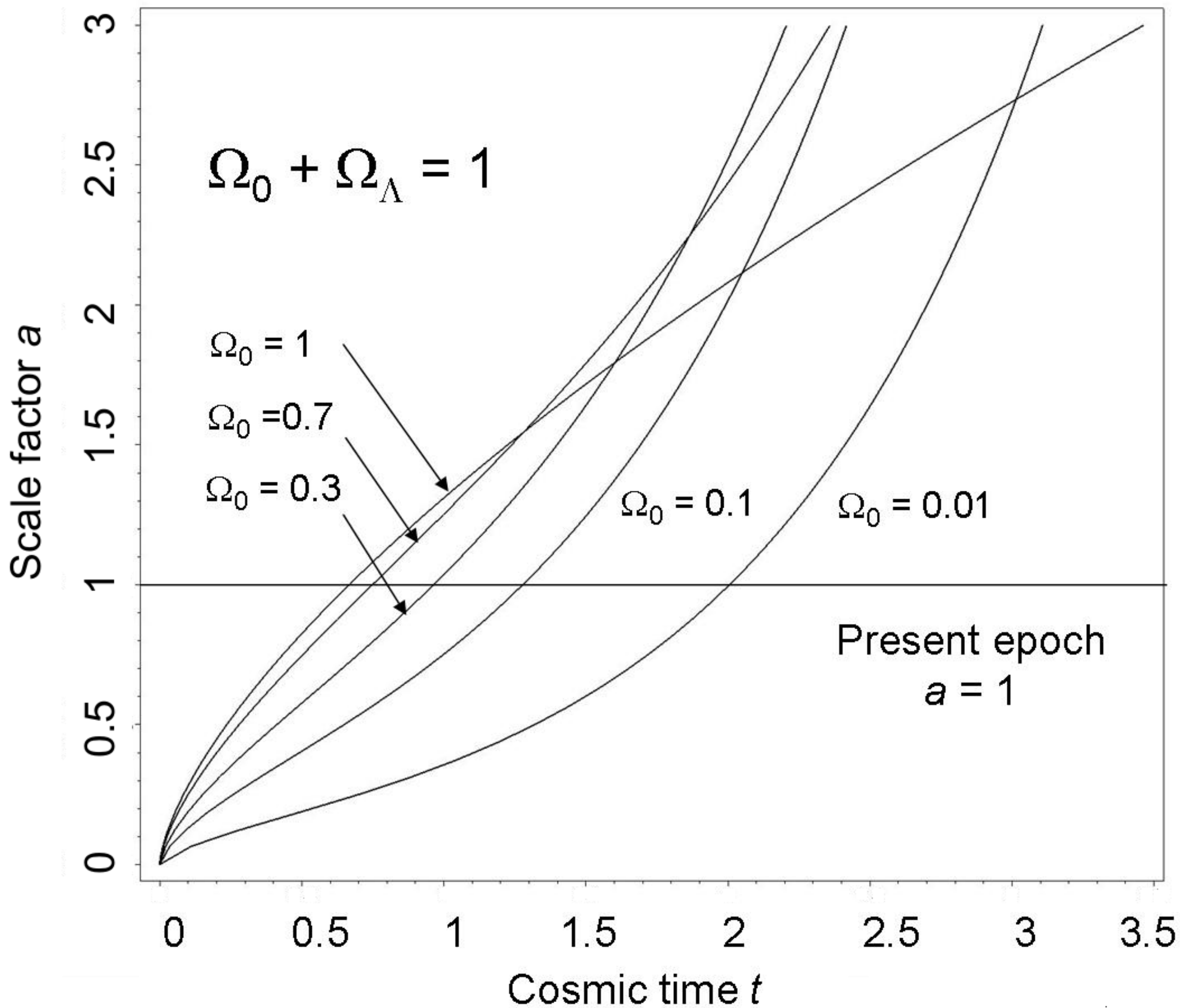


Classification of World Models with $\Lambda \neq 0$



The properties of the world models with non-zero cosmological constant are conveniently summarised in a plot of Ω_0 against $\Omega_0 + \Omega_\Lambda$ presented by Carroll, Press and Turner.

The Dynamics of World Models with $\Omega_0 + \Omega_\Lambda = 1$



The dynamics of spatially flat world models, $\Omega_0 + \Omega_\Lambda = 1$, with different combinations of Ω_0 and Ω_Λ. The abscissa is plotted in units of H_0^{-1}.

Estimating the Value of Ω_Λ

In their review of the problem of the cosmological constant, Carroll, Press and Turner described how a theoretical value of Ω_Λ could be estimated using simple concepts from quantum field theory. They found the mass density of the repulsive field to be $\varrho_{\rm V} = 10^{95}$ kg m^{-3}, about 10^{120} times greater than permissable values at the present epoch which correspond to $\rho_{\rm V} \leq 10^{-27}$ kg m^{-3}.

Heisenberg's Uncertainty Principle states that a virtual pair of particles of mass m can exist for a time $t \sim \hbar/mc^2$, corresponding to a maximum separation $x \sim \hbar/mc$. Hence, the typical density of the vacuum fields is $\rho \sim m/x^3 \approx c^3 m^4/\hbar^3$.

The mass density in the vacuum fields is unchanging with cosmic epoch and so, adopting the Planck mass for $m_{\rm Pl} = (hc/G)^{1/2} = 5.4 \times 18^{-8} = 3 \times 10^{19}$ GeV, the mass density corresponds to about 10^{97} kg m^{-3}. This is quite a problem. If the inflationary picture of the very early Universe is taken seriously, this is exactly the type of force which drove the inflationary expansion. Then, we have to explain why $\rho_{\rm V}$ decreased by a factor of about 10^{120} at the end of the inflationary era. In this context, 10^{-120} looks remarkably close to zero.

Radiation Dominated Universes

As discussed in the first lecture, the energy density per unit frequency range of black-body radiation is given by the Planck distribution

$$\varepsilon(\nu)\,\mathrm{d}\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT}-1}\,\mathrm{d}\nu\ . \tag{46}$$

The radiation temperature T_r varies with redshift as $T_\mathrm{r} = T_0(1+z)$ and the spectrum changes as

$$\begin{aligned}\varepsilon(\nu_1)\,\mathrm{d}\nu_1 &= \frac{8\pi h\nu_1^3}{c^3}[(e^{h\nu_1/kT_1}-1)]^{-1}\,\mathrm{d}\nu_1 \\ &= \frac{8\pi h\nu_0^3}{c^3}[e^{h\nu_0/kT_0}-1)^{-1}](1+z)^4\,\mathrm{d}\nu_0 \\ &= (1+z)^4\,\varepsilon(\nu_0)\,\mathrm{d}\nu_0\ . \end{aligned} \tag{47}$$

Thus, a black-body spectrum preserves its form as the Universe expands but the radiation temperature changes as $T_\mathrm{r} = T_0(1+z)$ and the frequency of each photon as $\nu = \nu_0(1+z)$. This is the same as the adiabatic expansion of a gas of photons. The ratio of specific heats γ for radiation and a relativistic gas in the ultrarelativistic limit is $\gamma = 4/3$ and so, in an adiabatic expansion, $T \propto V^{-(\gamma-1)} = V^{-1/3} \propto a^{-1}$, which is exactly the same as the above result.

Radiation Dominated Universes

The variations of p and ϱ with a can now be substituted into Einstein's field equations:

$$\ddot{a} = -\frac{4\pi G a}{3}\left(\varrho + \frac{3p}{c^2}\right) + \left[\tfrac{1}{3}\Lambda a\right] \ ;$$

$$\dot{a}^2 = \frac{8\pi G \varrho}{3} a^2 - \frac{c^2}{\Re^2} + \left[\tfrac{1}{3}\Lambda a^2\right] \ .$$

Therefore, setting the cosmological constant $\Lambda = 0$, we find

$$\ddot{a} = \frac{8\pi G \varepsilon_0}{3c^2}\frac{1}{a^3} \qquad \dot{a}^2 = \frac{8\pi G \varepsilon_0}{3c^2}\frac{1}{a^2} - \frac{c^2}{\Re^2} \ . \qquad (48)$$

At early epochs we can neglect the constant term $c^2/\Re^2$ and integrating

$$a = \left(\frac{32\pi G \varepsilon_0}{3c^2}\right)^{1/4} t^{1/2} \quad \text{or} \quad \varepsilon = \varepsilon_0 a^{-4} = \left(\frac{3c^2}{32\pi G}\right) t^{-2} \ . \qquad (49)$$

The dynamics of the radiation-dominated models, $a \propto t^{1/2}$, depend only upon the *total inertial mass density in relativistic and massless forms*. Thus, to determine the dynamics of the early Universe, we have to include all the massless and relativistic components in the total energy density. The force of gravity acting upon the sum of these determines the rate of deceleration of the early Universe.

The Cosmic Time–Redshift Relation

An important result for many aspects of astrophysical cosmology is the relation between cosmic time t and redshift z. Combining (34) and (36), we find

$$\dot{a} = H_0 \left[\Omega_0 \left(\frac{1}{a} - 1 \right) + \Omega_\Lambda (a^2 - 1) + 1 \right]^{1/2} . \tag{50}$$

Because $a = (1 + z)^{-1}$,

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -H_0 (1 + z) \left[(1 + z)^2 (\Omega_0 z + 1) - \Omega_\Lambda z (z + 2) \right]^{1/2} . \tag{51}$$

The cosmic time t measured from the Big Bang follows immediately by integration from $z = \infty$ to z,

$$t = \int_0^t \mathrm{d}t = -\frac{1}{H_0} \int_\infty^z \frac{\mathrm{d}z}{(1 + z)[(1 + z)^2 (\Omega_0 z + 1) - \Omega_\Lambda z (z + 2)]^{1/2}} . \tag{52}$$

Models with $\Omega_\Lambda = 0$

- For $\Omega_0 > 1$, we can write $x = (\Omega_0 - 1)a/\Omega_0 = (\Omega_0 - 1)/\Omega_0(1+z)$, and then

$$t(z) = \frac{\Omega_0}{H_0(\Omega_0 - 1)^{3/2}}\left[\sin^{-1} x^{1/2} - x^{1/2}(1-x)^{1/2}\right] . \quad (53)$$

- For $\Omega_0 < 1$, we write $y = (1 - \Omega_0)a/\Omega_0 = (1 - \Omega_0)/\Omega_0(1+z)$, and then

$$t(z) = \frac{\Omega_0}{H_0(1 - \Omega_0)^{3/2}}\left[y^{1/2}(1+y)^{1/2} + \sinh^{-1} y^{1/2}\right] . \quad (54)$$

- For large redshifts, $z \gg 1$, $\Omega_0 z \gg 1$, (53) and (54) reduce to

$$t(z) = \frac{2}{3H_0\Omega_0^{1/2}} z^{-3/2} . \quad (55)$$

- The present age of the Universe for the different world models is found by integrating from $z = 0$ to $z = \infty$. For the critical model $\Omega_0 = 1$, $t_0 = (2/3)H_0^{-1}$, for the empty model, $\Omega_0 = 0$, $t_0 = H_0^{-1}$ and for $\Omega_0 = 2$, $t_0 = 0.571\,H_0^{-1}$.

Models with $\Omega_\Lambda \neq 0$

The time-redshift relation for any of the models with finite Ω_Λ can be found by integration of (52). For models with zero curvature, there is a simple analytic solution for the cosmic time-redshift relation. From (36), the condition that the curvature of space is zero, $\Re \to \infty$, is $\Omega_0 + \Omega_\Lambda = 1$. Then, from (52),

$$t = \int_0^t \mathrm{d}t = -\frac{1}{H_0}\int_\infty^z \frac{\mathrm{d}z}{(1+z)[\Omega_0(1+z)^3 + \Omega_\Lambda]^{1/2}} \, . \tag{56}$$

The cosmic time–redshift relation becomes

$$t = \frac{2}{3H_0\Omega_\Lambda^{1/2}} \ln\left(\frac{1+\cos\theta}{\sin\theta}\right) \quad \text{where} \quad \tan\theta = \left(\frac{\Omega_0}{\Omega_\Lambda}\right)^{1/2}(1+z)^{3/2} \, . \tag{57}$$

The present age of the Universe follows by setting $z = 0$

$$t_0 = \frac{2}{3H_0\Omega_\Lambda^{1/2}} \ln\left[\frac{1+\Omega_\Lambda^{1/2}}{(1-\Omega_\Lambda)^{1/2}}\right] \, . \tag{58}$$

For example, if $\Omega_\Lambda = 0.9$ and $\Omega_0 = 0.1$, the age of the world model would be $1.28H_0^{-1}$. For the popular world model with $\Omega_0 = 0.3$ and $\Omega_\Lambda = 0.7$, the age of the Universe is $0.964H_0^{-1}$, remarkably close to H_0^{-1}.

Distance Measures as a Function of Redshift

We can now complete our programme of finding expressions for the comoving radial distance coordinate r and the distance measure D. We recall that the increment of comoving radial distance coordinate distance is

$$\mathrm{d}r = -\frac{c\,\mathrm{d}t}{a(t)} = -c\,\mathrm{d}t(1+z)\ . \tag{59}$$

From (51),

$$\mathrm{d}r = -\frac{c\,\mathrm{d}t}{a} = \frac{c}{H_0}\frac{\mathrm{d}z}{[(1+z)^2(\Omega_0 z+1) - \Omega_\Lambda z(z+2)]^{1/2}}\ . \tag{60}$$

and so, integrating from redshift 0 to z, we find the expression for r:

$$r = \frac{c}{H_0}\int_0^z \frac{\mathrm{d}z}{[(1+z)^2(\Omega_0 z+1) - \Omega_\Lambda z(z+2)]^{1/2}}\ . \tag{61}$$

Then, we can find the distance measure D by evaluating $D = \Re\sin(r/\Re)$, where $\Re$ is given by (36).

Models with $\Omega_\Lambda = 0$

Integrating (61) with $\Omega_\Lambda = 0$ and $\Omega_0 > 1$, we find

$$r = \frac{c}{H_0}\int_0^z \frac{\mathrm{dz}}{(1+z)(\Omega_0 z+1)^{1/2}} \tag{62}$$

$$= \frac{2c}{H_0(\Omega_0-1)^{1/2}}\left[\tan^{-1}\left(\frac{\Omega_0 z+1}{\Omega_0-1}\right)^{1/2} - \tan^{-1}(\Omega_0-1)^{-1/2}\right] . \tag{63}$$

If $\Omega_0 < 1$, the inverse tangents are replaced by inverse hyperbolic tangents. After some further straightforward algebra, we find that

$$D = \frac{2c}{H_0\Omega_0^2(1+z)}\left\{\Omega_0 z + (\Omega_0-2)[(\Omega_0 z+1)^{1/2}-1]\right\} . \tag{64}$$

This is the famous formula first derived by Mattig. Although the integral has been found for the case of spherical geometry, it turns out that the formula is correct for all values of Ω_0. In the limit of the empty, or Milne, world model, $\Omega_0 = 0$, (64) becomes

$$D = \frac{cz}{H_0}\frac{\left(1+\frac{z}{2}\right)}{(1+z)} . \tag{65}$$

Space-time Diagrams for the Standard World Models

Let us summarise the various times and distances used in cosmological analyses.

Comoving radial distance coordinate In terms of cosmic time and scale factor, the comoving radial distance coordinate r is defined to be

$$r = \int_t^{t_0} \frac{c\,\mathrm{d}t}{a} = \int_a^1 \frac{c\,\mathrm{d}a}{a\dot{a}} \,. \tag{66}$$

Proper radial distance coordinate We run up against the same problems we encountered in defining the comoving radial distance coordinate, in that it only makes sense to define distances at a particular cosmic epoch t. Therefore, we *define* the proper radial distance r_{prop} to be the comoving radial distance coordinate projected back to the epoch t, that is

$$r_{\mathrm{prop}} = a \int_t^{t_0} \frac{c\,\mathrm{d}t}{a} = a \int_a^1 \frac{c\,\mathrm{d}a}{a\dot{a}} \,. \tag{67}$$

Space-time Diagrams for the Standard World Models

Particle horizon The particle horizon r_{H} is defined as the maximum proper distance over which there can be causal communication at the epoch t

$$r_{\mathrm{H}} = a \int_0^t \frac{c\,\mathrm{d}t}{a} = a \int_0^a \frac{c\,\mathrm{d}a}{a\dot{a}} \, . \tag{68}$$

Event horizon The event horizon r_{E} is defined as the greatest proper radial distance an object can have if it is ever to be observable by an observer who observes the Universe at cosmic time t_1.

$$r_{\mathrm{E}} = a \int_{t_1}^{t_{\mathrm{max}}} \frac{c\,\mathrm{d}t}{a(t)} = a \int_{a_1}^{a_{\mathrm{max}}} \frac{c\,\mathrm{d}a}{a\dot{a}} \, . \tag{69}$$

Space-time Diagrams for the Standard World Models

Cosmic time Cosmic time t is defined to be time measured by a fundamental observer who reads time on a standard clock.

$$t = \int_0^t \mathrm{d}t = \int_0^a \frac{\mathrm{d}a}{\dot{a}} \,. \tag{70}$$

Conformal time The *conformal time* τ is similar to the definition of comoving radial distance coordinate. Time intervals are projected forward to present epoch

$$\mathrm{d}t_{\mathrm{conf}} = \mathrm{d}\tau = \frac{\mathrm{d}t}{a} \,. \tag{71}$$

At any epoch, the conformal time has value

$$\tau = \int_0^t \frac{\mathrm{d}t}{a} = \int_0^a \frac{\mathrm{d}a}{a\dot{a}} \,. \tag{72}$$

It follows that, in a space-time diagram in which comoving radial distance coordinate is plotted against conformal time, the particle horizon is a straight line with slope equal to the speed of light.

The Past Light Cone

This topic requires a little care. First, because of the assumptions of isotropy and homogeneity, Hubble's linear relation $v = H_0 r$ applies at the present epoch *to recessions speeds which exceed the speed of light*, where r is the radial comoving distance coordinate. Recall how we defined r. The fundamental observers measured increments of distance Δr at the present epoch t_0. If we consider fundamental observers who are far enough apart, this speed can exceed the speed of light. There is nothing in this argument which contradicts the special theory of relativity – it is a geometric result because of the requirements of isotropy and homogeneity.

Consider the analogue for the expanding Universe of the surface of an expanding spherical balloon. As the balloon inflates, a linear velocity-distance relation is found on the surface of the sphere, not only about any point on the sphere, but also at arbitrarily large distances on its surface. At very large distances, the speed of separation can be greater than the speed of light, but there is no causal connection between these points – they are simply partaking in the uniform expansion of the underlying space-time geometry of the Universe.

The Past Light Cone

Consider the proper distance between two fundamental observers at some epoch t

$$r_{\mathrm{prop}} = a(t)r \ , \tag{73}$$

where r is comoving radial distance. Differentiating with respect to cosmic time,

$$\frac{\mathrm{d}r_{\mathrm{prop}}}{\mathrm{d}t} = \dot{a}r + a\frac{\mathrm{d}r}{\mathrm{d}t} \ . \tag{74}$$

The first term on the right-hand side represents the motion of the substratum and, at the present epoch, becomes $H_0 r$. Consider, for example, the case of a very distant object in the critical world model, $\Omega_0 = 1, \Omega_\Lambda = 0$. As a tends to zero, the comoving radial distance coordinates tends to $r = 2c/H_0$. Therefore, the local rest frame of objects at these large distances moves at twice the speed of light relative to our local frame of reference *at the present epoch*. At the epoch at which the light signal was emitted along our past light cone, the recessional velocity of the local rest frame $v_{\mathrm{rec}} = \dot{a}r$ was greater than this value, because $\dot{a} \propto a^{-1/2}$.

The Past Light Cone

The second term on the right-hand side of (74) corresponds to the velocity of peculiar motions in the local rest frame at r, since it corresponds to changes of the comoving radial distance coordinate. The element of proper radial distance is $a\mathrm{d}r$ and so, if we consider a light wave travelling along our past light cone towards the observer at the origin, we find

$$\boxed{v_{\mathrm{tot}} = \dot{a}r - c\,.} \tag{75}$$

This is the key result which defines the propagation of light from the source to the observer in space-time diagrams for the expanding Universe.

We can now plot the trajectories of light rays from their source to the observer at t_0. The proper distance from the observer at $r = 0$ to the past light cone r_{PLC} is

$$r_{\mathrm{PLC}} = \int_0^t v_{\mathrm{tot}}\,\mathrm{d}t = \int_0^a \frac{v_{\mathrm{tot}}\,\mathrm{d}a}{\dot{a}}\,. \tag{76}$$

The Past Light Cone

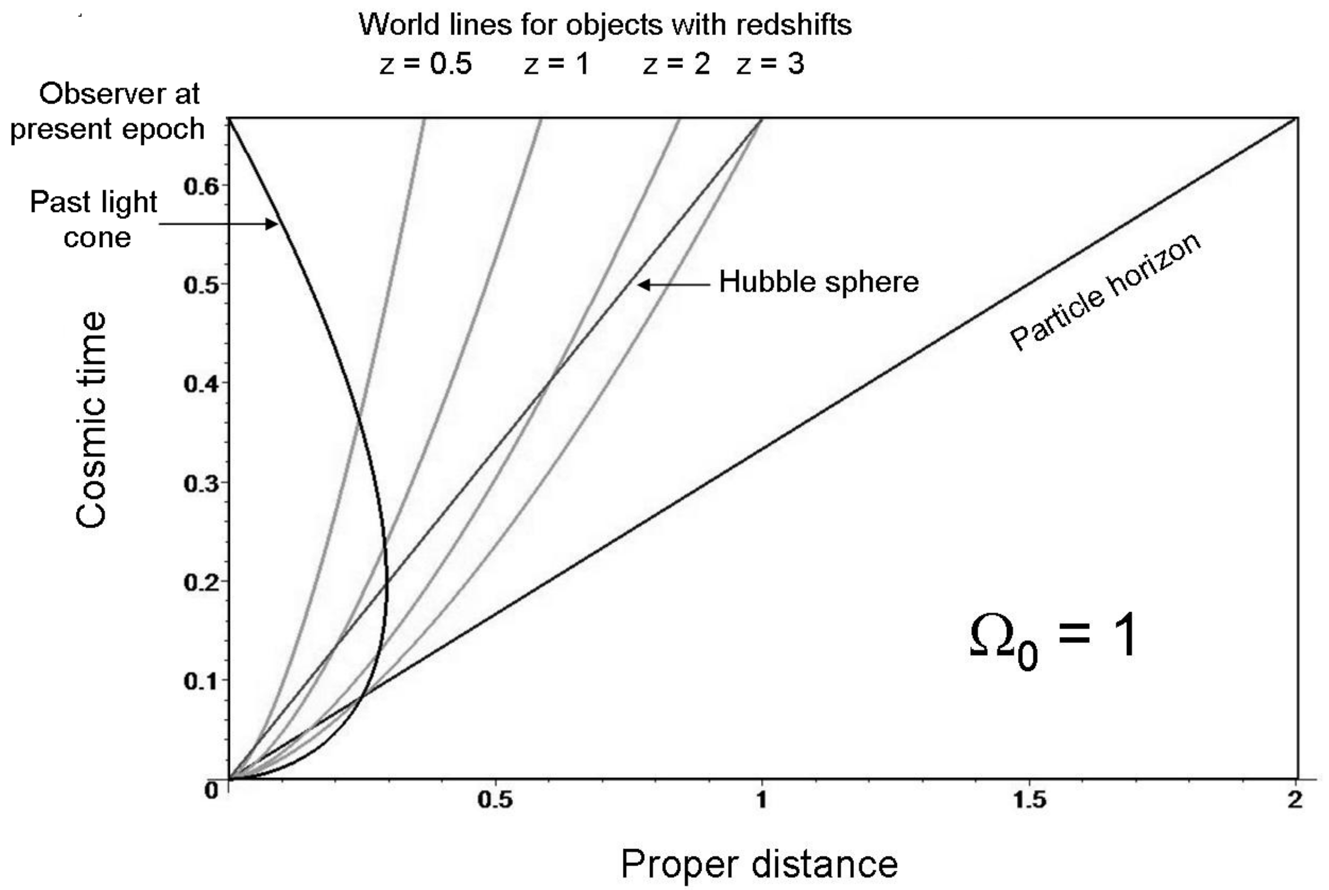
Notice that, initially the light rays from distant objects are propagating away from the observer – this is because the local isotropic cosmological rest frame is moving away from the observer at $r = 0$ at a speed greater than that of light. The light waves are propagated to the observer at the present epoch through local inertial frames which expand with progressively smaller velocities until they cross the *Hubble sphere* at which the recession velocity of the local frame of reference is the speed of light. The definition of the radius of the Hubble sphere $r_{\rm HS}$ at epoch t is thus given by

$$c = H(t)\, r_{\rm HS} = \frac{\dot{a}}{a} r_{\rm HS} \qquad \text{or} \qquad r_{\rm HS} = \frac{ac}{\dot{a}}\,. \tag{77}$$

Note that $r_{\rm HS}$ is a proper radial distance. From this epoch onwards, propagation is towards the observer until, as $t \to t_0$, the speed of propagation towards the observer is the speed of light.

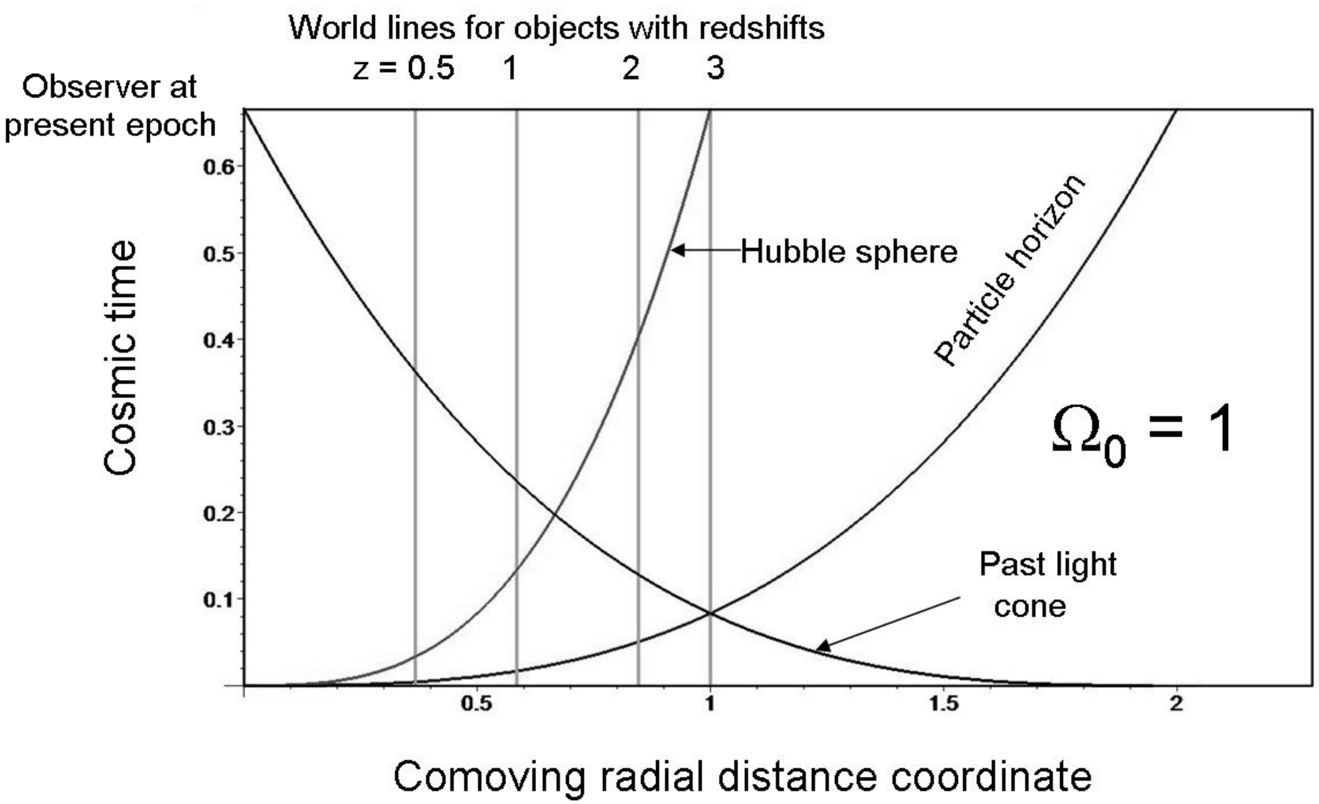
It is simplest to illustrate how the various scales change with time in specific examples of standard cosmological models. We consider first the critical world model and then our reference Λ model.

Space-Time Diagram
Cosmic Time vs. Proper Distance



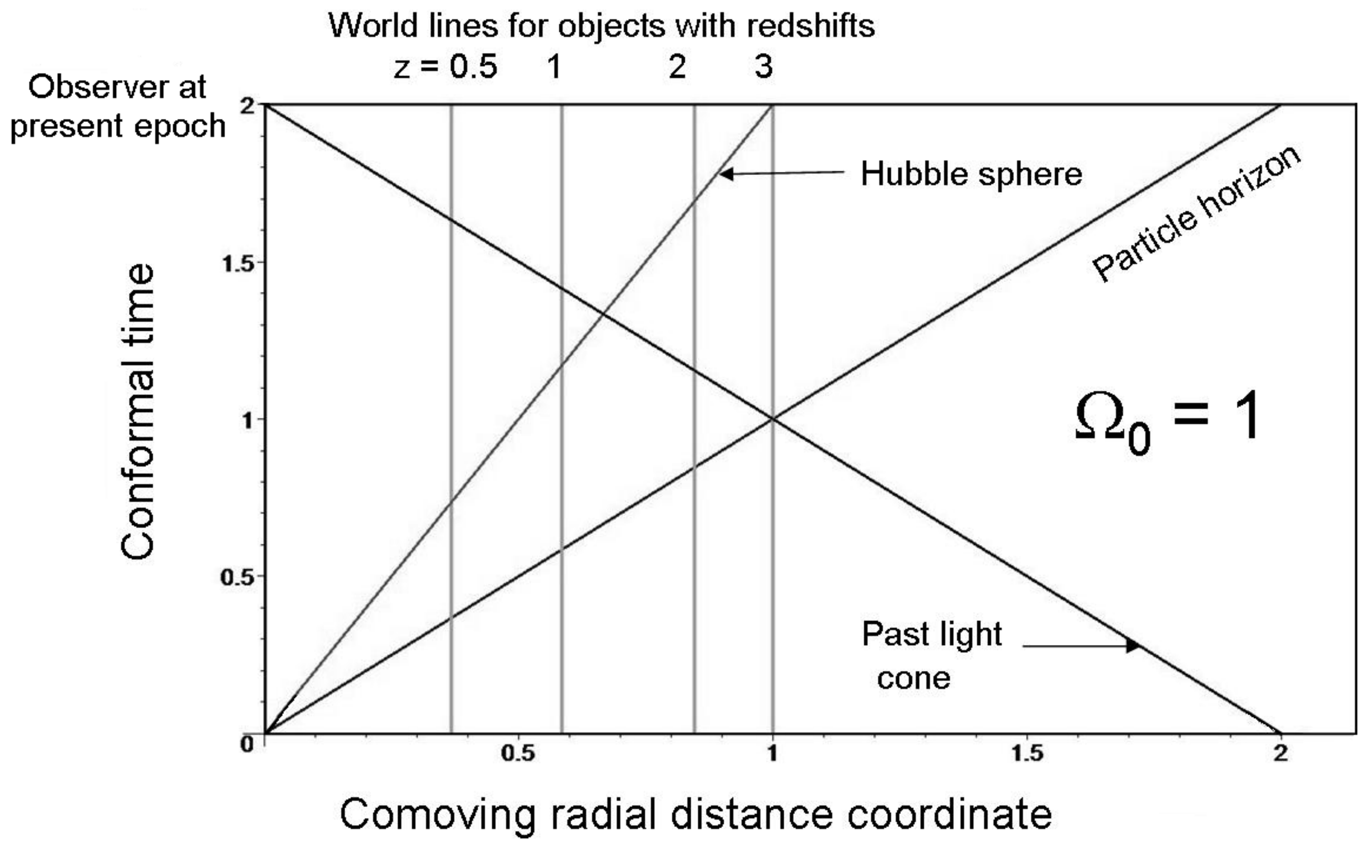
The times and distances are measured in units of H_0^{-1} and c/H_0 respectively.

Space-Time Diagram
Cosmic Time vs. Comoving Distance Coordinate



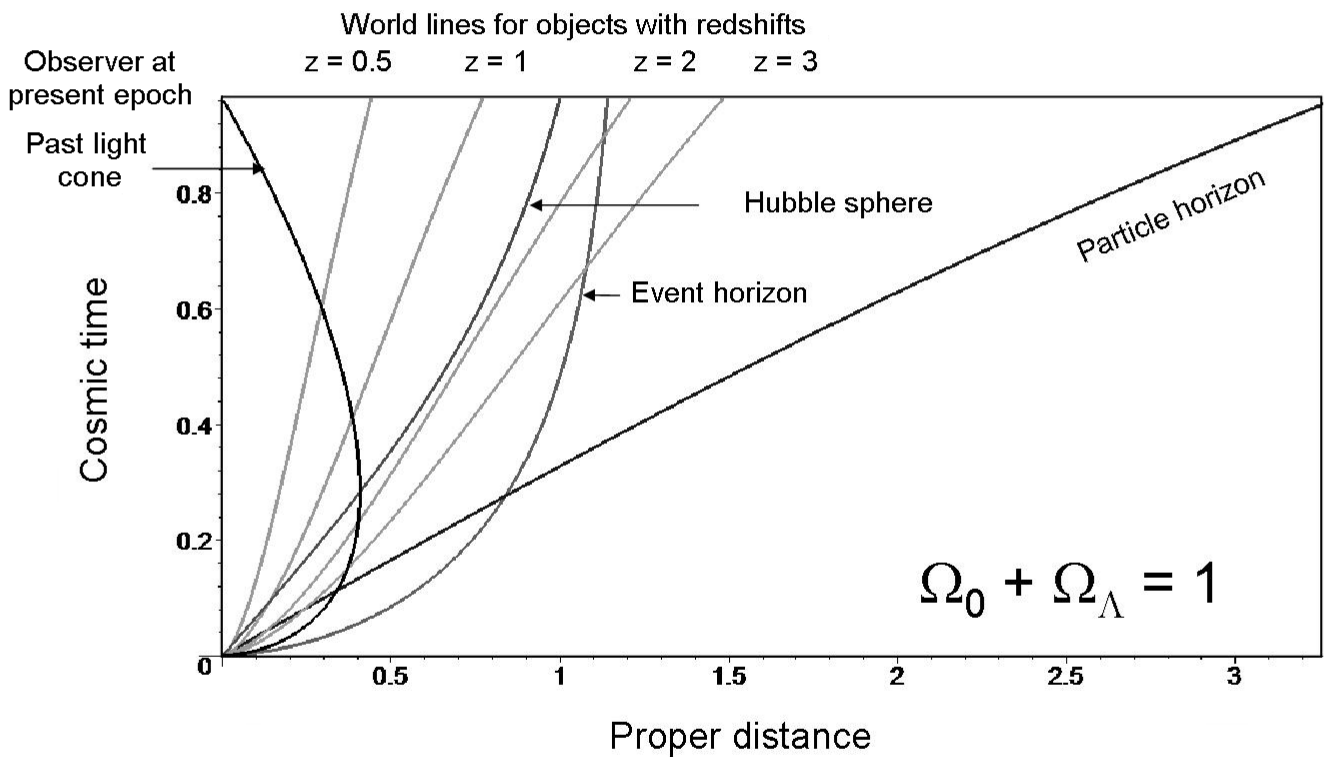
The times and distances are measured in units of H_0^{-1} and c/H_0 respectively.

Space-Time Diagram
Conformal Time vs. Comoving Distance Coordinate



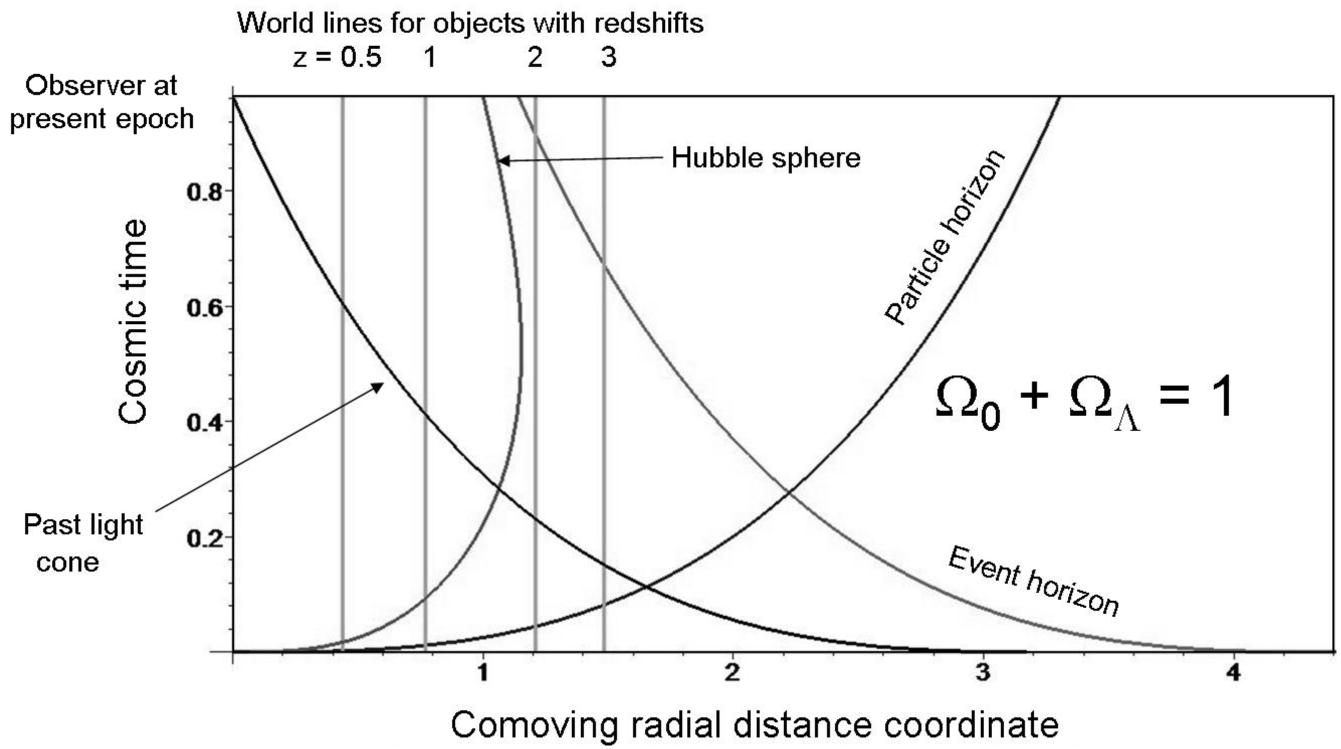
The times and distances are measured in units of H_0^{-1} and c/H_0 respectively.

Space-Time Diagram
Cosmic Time vs. Proper Distance



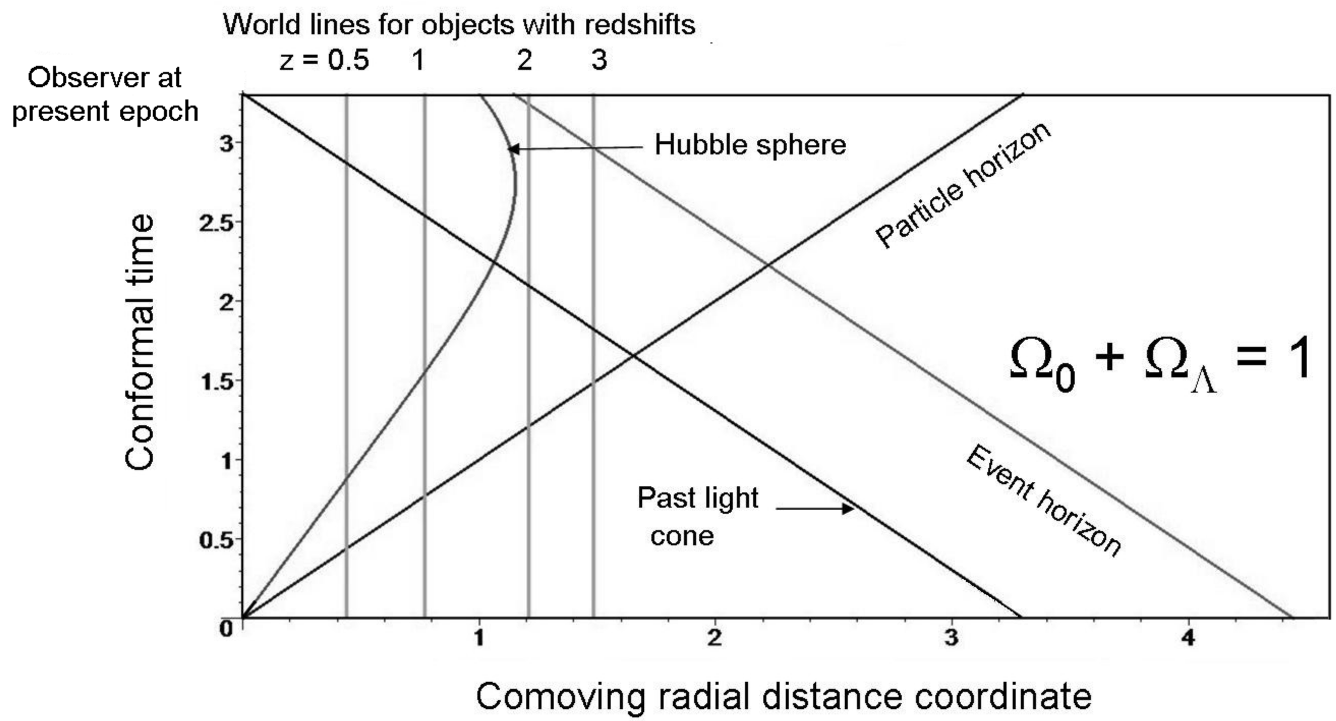
$\Omega_0 = 0.3$. The times and distances are measured in units of H_0^{-1} and c/H_0 respectively.

Space-Time Diagram
Cosmic Time vs. Comoving Distance Coordinate



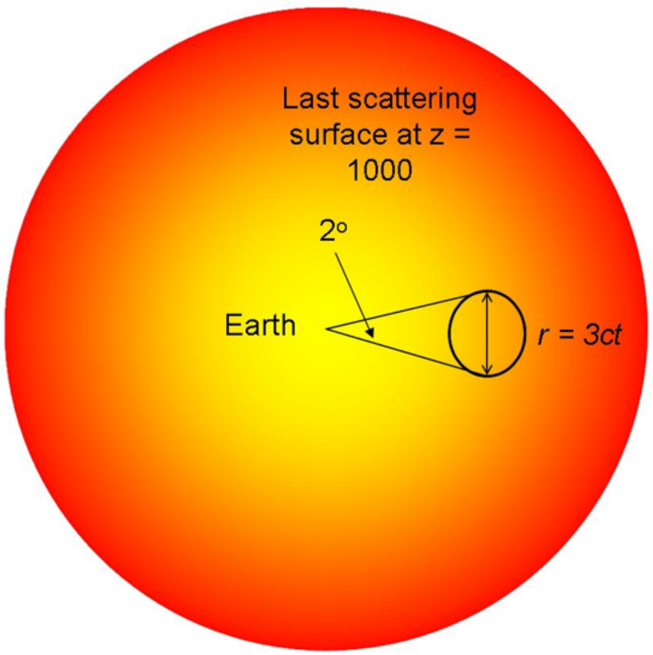
$\Omega_0 = 0.3$. The times and distances are measured in units of H_0^{-1} and c/H_0 respectively.

Space-Time Diagram
Conformal Time vs. Comoving Distance Coordinate



$\Omega_0 = 0.3$. The times and distances are measured in units of H_0^{-1} and c/H_0 respectively.

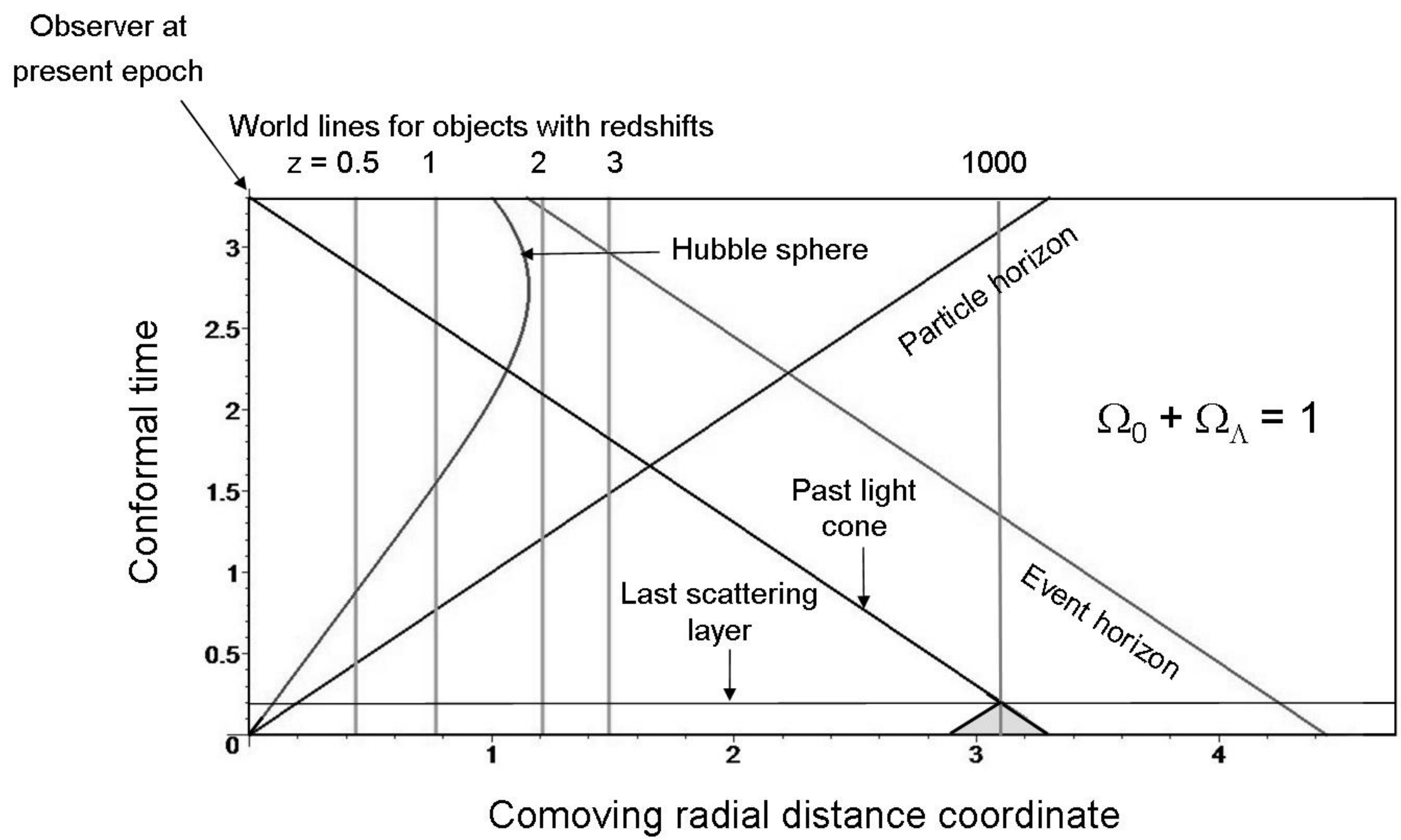
The Horizon Problem



'Why is the Universe so isotropic?' At earlier cosmological epochs, the particle horizon $r \sim ct$ encompassed less and less mass and so the scale over which particles could be causally connected became smaller and smaller. On the last scattering surface at $z \approx 1,500$, the particle horizon corresponds to an angle $\theta \approx 2^\circ$ on the sky. How did opposite sides of the sky know they had to have the same properties within one part in 10^5?

The Horizon Problem

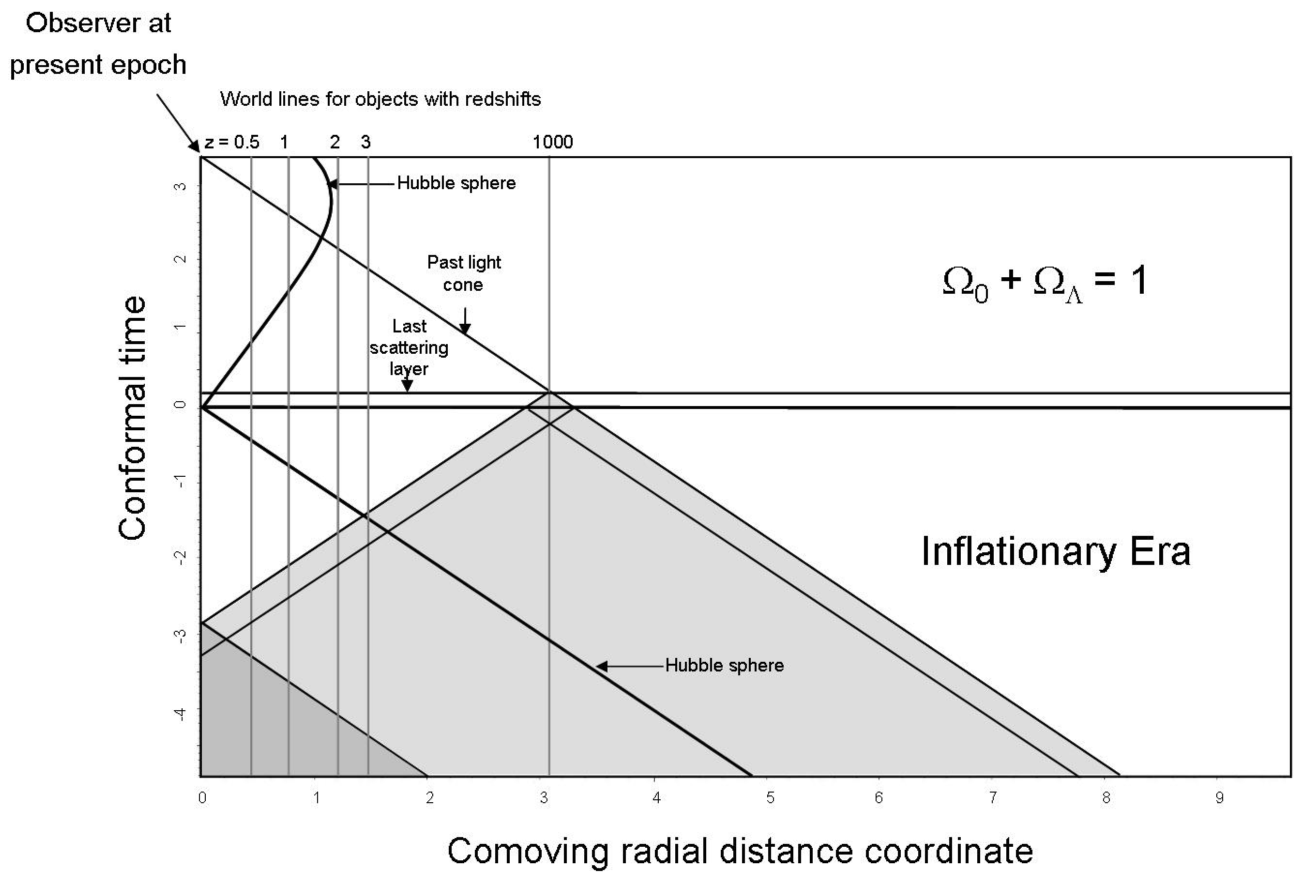
In this version of the conformal diagram, we have included the epoch of recombination and the past light cone from that epoch back to the initial singularity.



The times and distances are measured in units of H_0^{-1} and c/H_0 respectively.

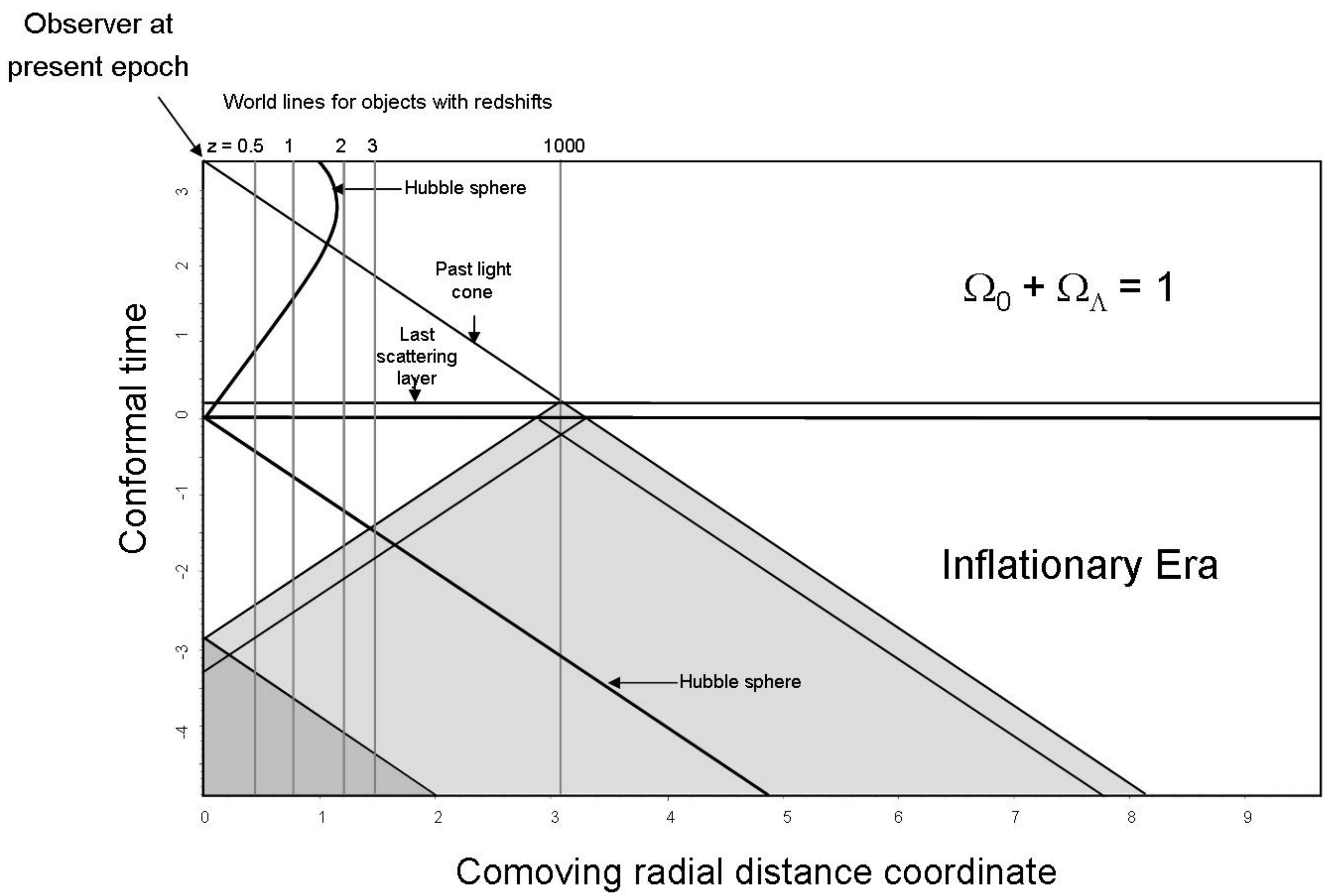
The Inflationary Picture

The inflationary picture solves the horizon and flatness problems by assuming there was a period of exponential growth of the scale factor in the very early Universe.



In the extended conformal time diagram, the time coordinate is set to zero at the end of the inflationary era at, say, 10^{-32} s and evolution of the Hubble sphere and the past light cone at recombination extrapolated back to the inflationary era.

The Inflationary Picture



- The particle horizon is the maximum distance over which causal contact could have been made from the time of the singularity to a given epoch. The radius of the Hubble sphere is the distance of causal contact at a particular epoch, $c = Hr$.
- The point at which the Hubble sphere crosses the comoving radial distance coordinate of the last scattering surface, exactly corresponds to the time when the past light cones from opposite directions on the sky touch at conformal time -3.

The Inflationary Picture

Because any object preserves its comoving radial distance coordinate for all time, in the early Universe, objects lie within the Hubble sphere, but during the inflationary expansion, they pass through it and remain outside it for the rest of the inflationary expansion. Only when the Universe transforms back into the standard Friedman model does the Hubble sphere begin to expand again and objects can then 're-enter the horizon'. This behaviour occurs for all scales and masses of interest in understanding the origin of structure in the present Universe.

Since causal connection is no longer possible on scales greater than the Hubble sphere, objects 'freeze out' when they pass through the Hubble sphere during the inflationary era, but they come back in again and regain causal contact when they recross the Hubble sphere.

The inflationary expansion also drives the geometry to a flat geometry since its radius of curvature $\Re \to \infty$, solving the fine-tuning problem.