Physics of Primary CMB Anisotropies

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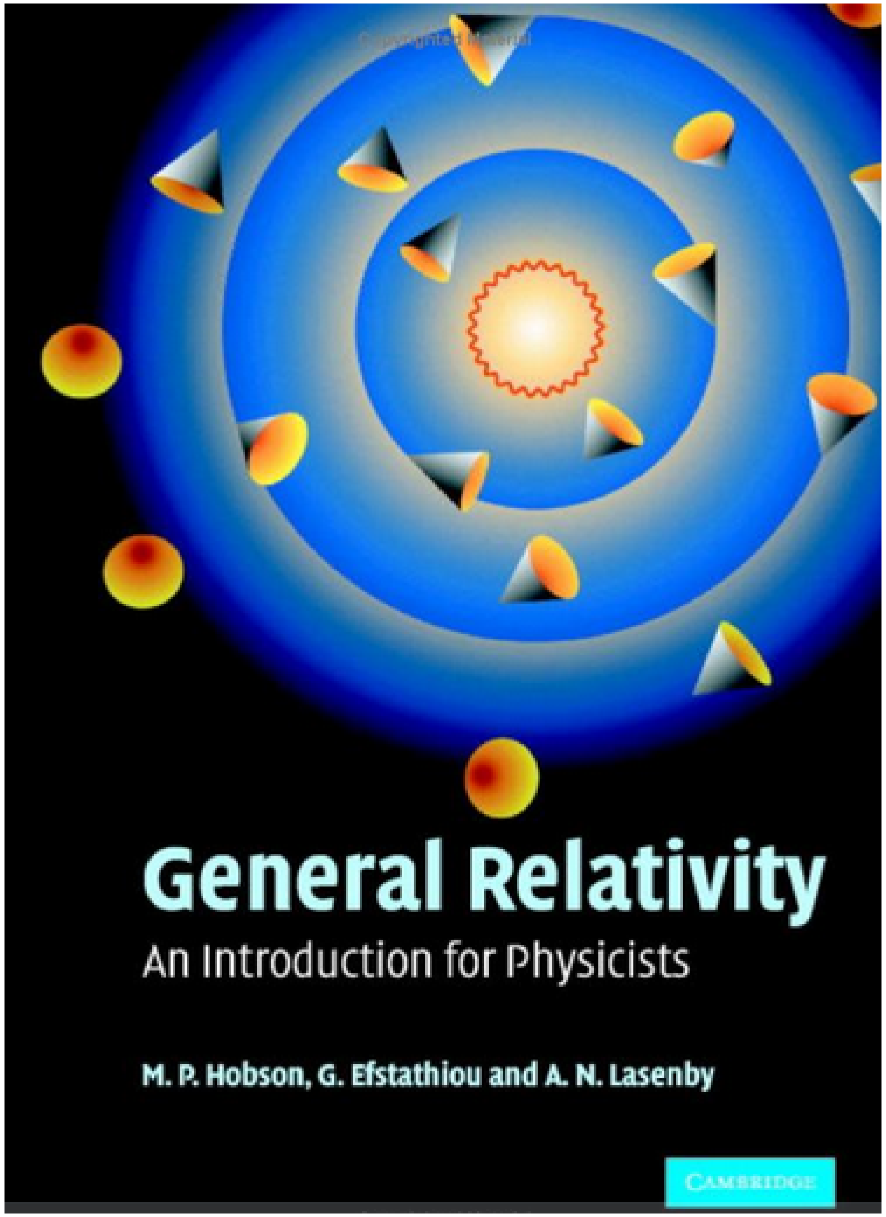
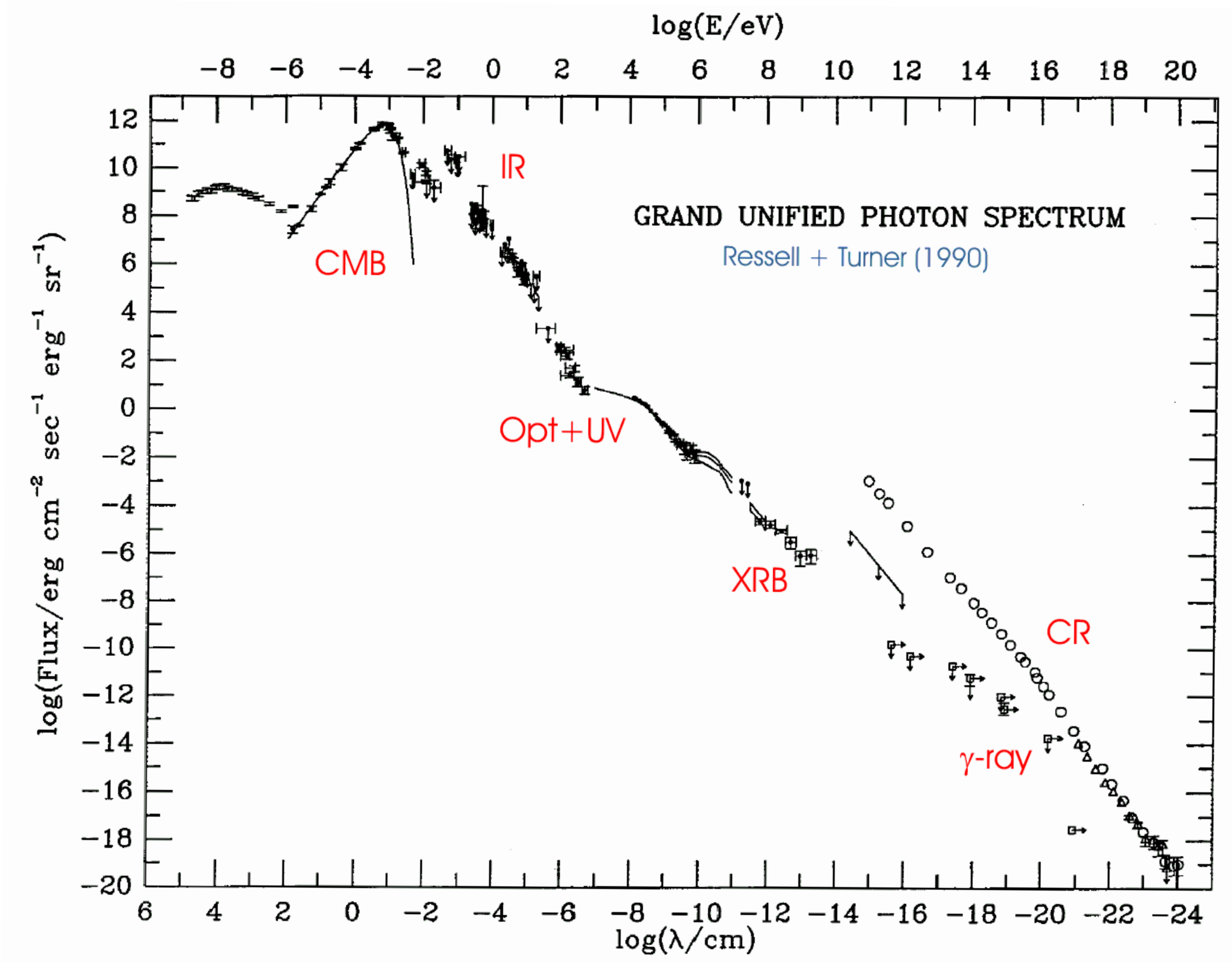
ACKNOWLEDGEMENTS/BOOKS

- Thanks to Anthony Challinor for help with some of the later slides
- In terms of dealing with perturbations, lots of conventions for symbols and quantities exist
- I have mainly followed those in 'The Primordial Density Perturbation', by D. Lyth and A. Liddle, CUP (2009)
- Very useful book
- Also note Hobson, Efstathiou & Lasenby (opposite)
- Builds up GR from scratch, but also gets through to e.g. development of scalar field perturbations in inflation



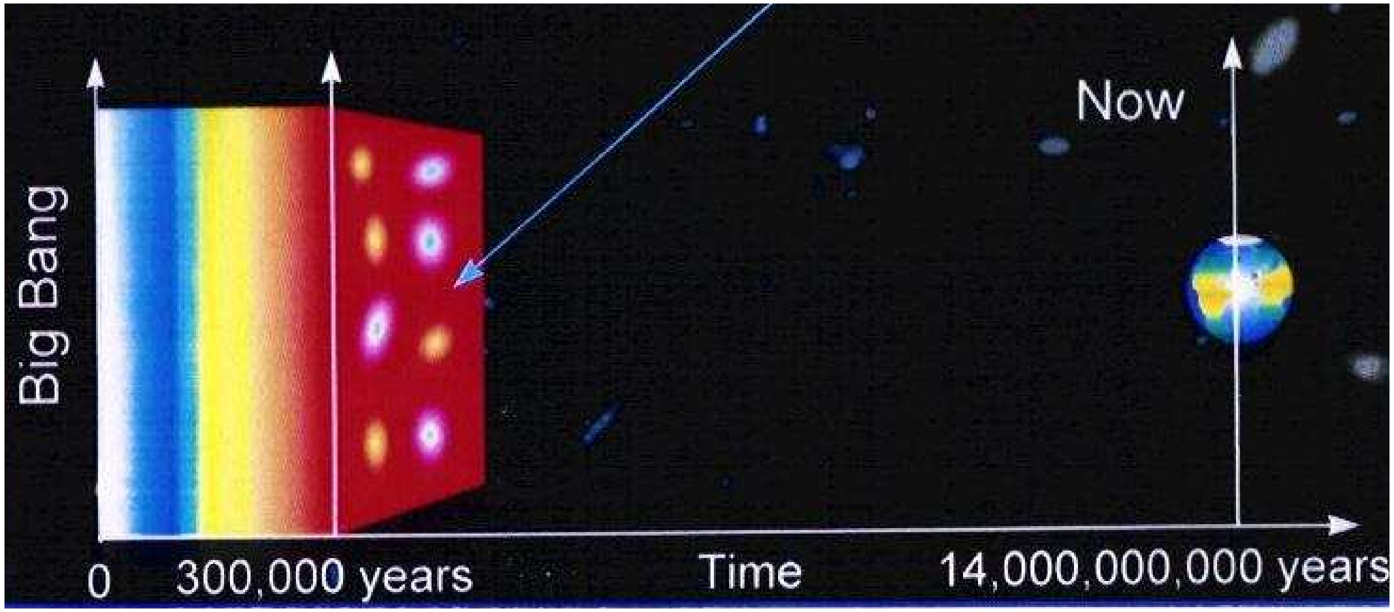
The Grand Unified Photon Spectrum



THE CMB AS PART OF THE GUPS

- The blackbody nature of the CMB immediately tells one it will have early universe information encoded in it
  - (Note perfection of CMB blackbody exceeded reference bb on COBE satellite after just 9 minutes of data!)
- Currently not clear that any of the other regions of the spectrum represent a real diffuse spectrum rather than discrete (but numerous) point sources
- Also dust and absorption of various kinds is a real problem for most of the near infrared, optical and UV wavelength ranges (e.g Ressell and Turner, Comments on Astrophysics, **14**, 323 (1990) is still a useful survey), though CIB from Planck is also starting to become useful

THE COSMIC MICROWAVE BACKGROUND



- The Cosmic Microwave Background (CMB) was emitted at about 300,000 years after the big bang and has been propagating to us ever since
- Think about 90% of the photons make it straight to us, telling us about the physics at the time of recombination
- Rest carry imprints of what has happened on the way
- But when emitted also has encoded in it information dating from about 10^{-36} seconds after the big bang

THE COSMIC MICROWAVE BACKGROUND (CONTD.)

- Huge advances in technology in past few years, are enabling us to measure all 3 of these aspects with rapidly increasing precision
- Has finally ushered us into an era of ‘precision cosmology’ (but also deep mysteries)
- Will try in these 2 lectures to give an overview of the physics that goes into producing the anisotropies
- The exciting bit about what happens at 10^{-36} seconds belongs to inflation (David Seery), and current observations and implications belongs to Paolo de Bernardis/Rafa Rebola/Lucio Piccirillo
- However, as an intellectual endeavour, think the prediction of the structure of the CMB power spectrum, relying on detailed application of GR perturbation theory, and atomic physics, is one of the pinnacles of modern science (first predicted about 1968?, first seen about 1998?) and belongs right up there with the Higgs

A USEFUL RESULT

- We're going to start with the Liouville theorem in curved spacetime
- Sounds a bit abstract, but very useful, and gives us a quick route to understanding important aspects of the physics of CMB anisotropies
- So the theorem is this: At each point on the worldline of a particle travelling along a geodesic in spacetime, the local phase space density of surrounding particles is (a) a local Lorentz invariant, and (b) constant along the worldline.
- (The Liouville bit is really part (b). Part (a) is automatic, as we'll see.)
- The theorem as stated applies to any type of particle, but applied to photons, then since local phase space density turns out to be I_ν/ν^3, where I_ν is Intensity, then we can rephrase as:
- At each point on the worldline of a photon travelling along a geodesic in spacetime, I_ν/ν^3 is (a) a local Lorentz invariant, and (b) constant along the worldline.

A USEFUL RESULT

- The proof that the phase space density is a local Lorentz invariant boils down to the following.
- Saying that the phase space density is $f(p_x, p_y, p_z, x, y, z)$ tells you that the number of particles with momenta in the range p_x to $p_x + dp_x$, p_y to $p_y + dp_y$ and p_z to $p_z + dp_z$, and with positions in the range x to $x + dx$, y to $y + dy$ and z to $z + dz$ is

$$f(p_x, p_y, p_z, x, y, z) dp_x dp_y dp_z dx dy dz$$

- Next we observe that if we travel with Lorentz factor γ in the direction x say, then relative to the rest-frame case, the infinitesimal volume factor is shrunk by a factor $1/\gamma$ (by Lorentz contraction in the boost direction) and dp_x is increased by a factor γ. (If $p_x = m \sinh \alpha$, then $dp_x = m \cosh \alpha d\alpha = m\gamma d\alpha$.)
- This is for boosts, and spatial rotations are obviously ok.
- For I_ν/ν^3 being phase space density, remember momenta are $h\nu/c$ and definition of intensity

Applications to black body radiation

- Let's apply this to black body radiation. This has

$$I_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}$$

- Thus the invariant I_ν/ν^3 being constant is the same as $h\nu/kT$ being constant, in other words, for b.b. radiation $T_{\rm rad}$ just tracks the photon frequency

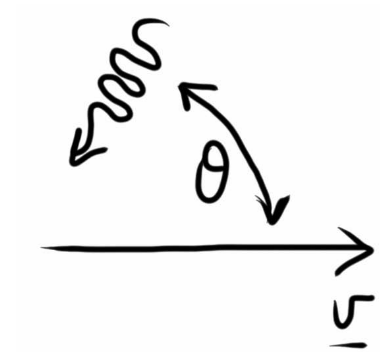
- Knowing this gives great simplifications in working out what we see in various contexts

- E.g. for cosmological expansion, since $\nu \propto 1+z$ (by definition), we see $T_{\rm rad} \propto (1+z)$ also (in the b.b. case)

- Also, remultiplying the invariant up by ν^3, we see we have proved radiation that is initially of b.b. form, remains that way whilst propagating (as long as there are no collisions)

- So it manages to do this, even though non-interacting (and proof is much quicker than any other route)

APPLICATIONS TO BLACK BODY RADIATION



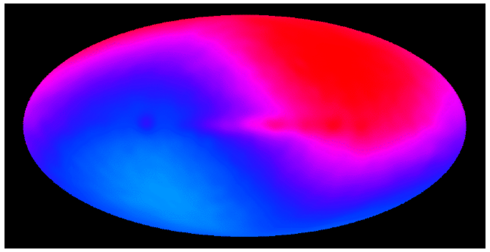
- Another case: Suppose we are moving w.r.t. a black body radiation field.
- What do we see?
- So just need to find out how photon frequency is changed relative to what we would get with no velocity
- If p is the photon 4-momentum, v_1 the 4-velocity of the moving observer and v_0 is the 4-velocity of an observer of a frame in which the b.b. radiation is isotropic, with temperature T_0 say, easy to show

$$\frac{p \cdot v_1}{p \cdot v_0} = \frac{1 + \beta \cos\theta}{\sqrt{1 - \beta^2}}$$

where $\beta = v/c$. So we deduce

$$T_{\rm obs}(\theta) = \frac{1 + \beta \cos\theta}{\sqrt{1 - \beta^2}} T_0$$

APPLICATIONS TO BLACK BODY RADIATION (CONTD.)



- This is pretty much what we'd expect to see
- Get a dipole around the sky proportional to $\cos\theta$
- But the angle θ we're using here is measured in the frame in which the radiation is isotropic
- What do things look like in terms of the angle the moving observer would measure?
- So have to include the abberation formula

$$\cos(\theta') = \frac{\cos(\theta) + \beta}{1 + \beta\cos\theta}$$

- Solving for $\cos\theta$ and substituting, get

$$T_{\mathrm{obs}}(\theta') = \frac{\sqrt{1-\beta^2}}{1-\beta\cos\theta'}T_0$$

APPLICATIONS TO BLACK BODY RADIATION (CONTD.)

- Can get a clearer idea of what this means by expanding to 2nd order in β
- Find

$$T_{\rm obs}(\theta') \approx T_0(1 + \beta\cos\theta + \tfrac{1}{2}\beta^2\cos 2\theta)$$

- So the abberation has induced a quadrupole anisotropy, proportional to $(v/c)^2$
- Known as the kinematic quadrupole
- Interesting to compare current invariant-based method with the early calculations in this area
- E.g. Condon & Harwit (Phys.Rev.Lett., **20**, 1309 (1968)) gave first derivation of this effect, considering time dilation, transformation of solid angles, frequency transformation, aberration, transformation of detector area, etc, and got:

APPLICATIONS TO BLACK BODY RADIATION (CONTD.)

- So they had a frequency dependent effect (therefore not b.b. afterwards) actually going wrong way round at lower frequencies!
- First people to get it right were Peebles & Wilkinson, Phys.Rev. **174**, 2168 (1968)
- They also had to consider all the above effects and combine them to get result
- First to do it simple way using the invariant was Forman, Planet.Space.Sci., **18**, 25 (1970), in the context of the Compton-Getting effect in cosmic ray physics, where a dipole due to the Earth's motion had first been noticed in the 1930's
- So what do we actually see for the CMB?

The dimensionless function

$$f(x) = \frac{x^3}{e^x - 1}\left(\frac{x e^x}{e^x - 1} - 2\right)$$

is plotted in Fig. 1. At low frequencies it is negative, indicating that the detector receives less power from an approaching source than from a receding one. $f(x)$ becomes positive for $x > 1.60$ and has a strong maximum near $x = 4.5$.

In the case of the 3°K radiation field, almost all of the anisotropy signal power is found in the region around $\lambda = 1$ mm. The total anisotropy signal power that would be received by a bolometric detector of unit throughput because of the earth's motion with respect to the radiation is

$$P = \beta\left(\frac{4k^4T^4}{c^2h^3}\right)\int_0^\infty f(x)dx,$$

$$P = \beta \times 6 \times 10^{-3} \text{ erg/sec.}$$

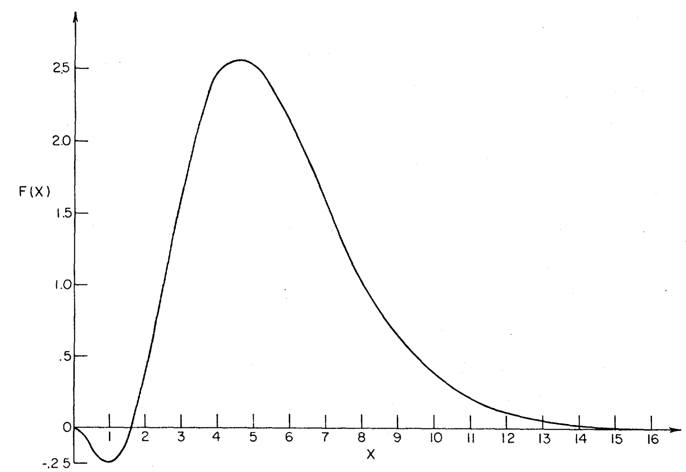
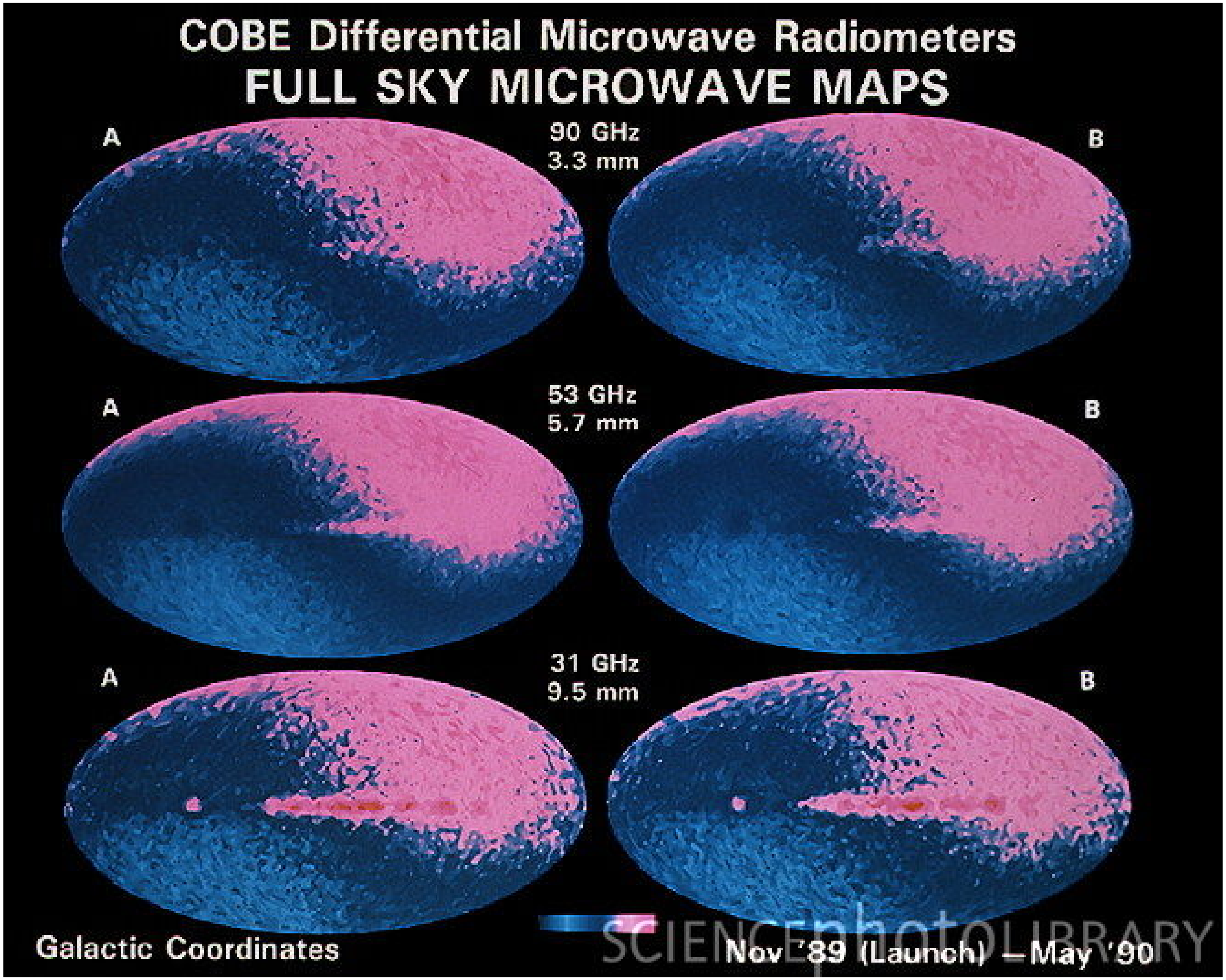


FIG. 1. Relative anisotropy signal strength $f(x)$ (dimensionless) as a function of $x = h\nu'/kT$.

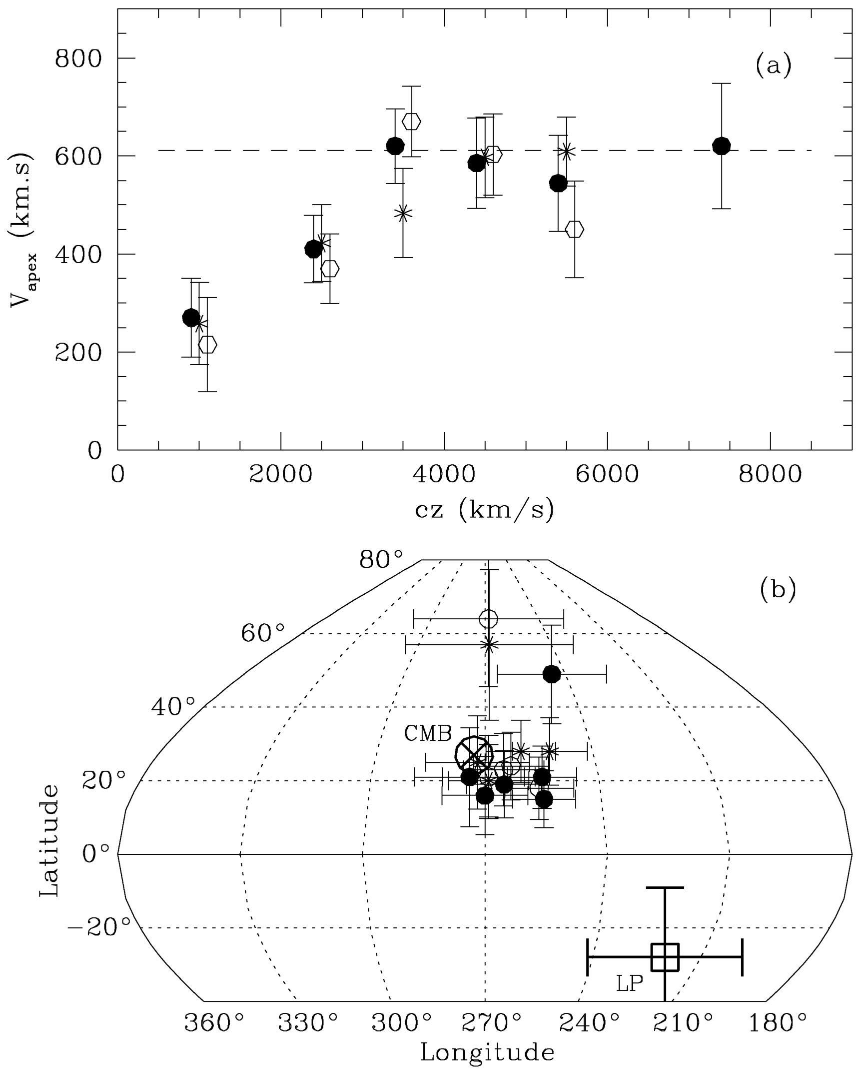
From Harwit & Condon (1968)

The CMB Dipole



The CMB Dipole

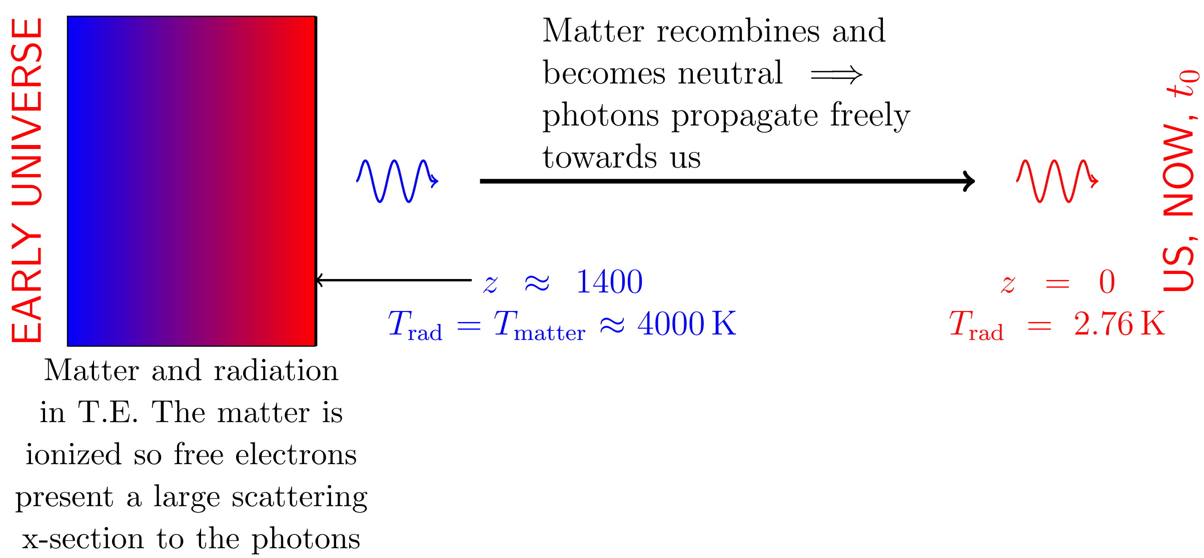
- As is well understood now, most if not all of this is due to the peculiar motion of the Earth/Sun/Galaxy/Local Group system
- There is indeed a kinematic quadrupole induced by this motion, and we have to be careful to subtract this in order to discover the primordial monopole
- Also we don't really know how much of the dipole might still be primordial —- presumably small since we get a fairly good alignment of CMB dipole with that from peculiar velocity surveys, but still an open question



From Giovanelli et al., 1998 (Ap.J., **505**, L91)

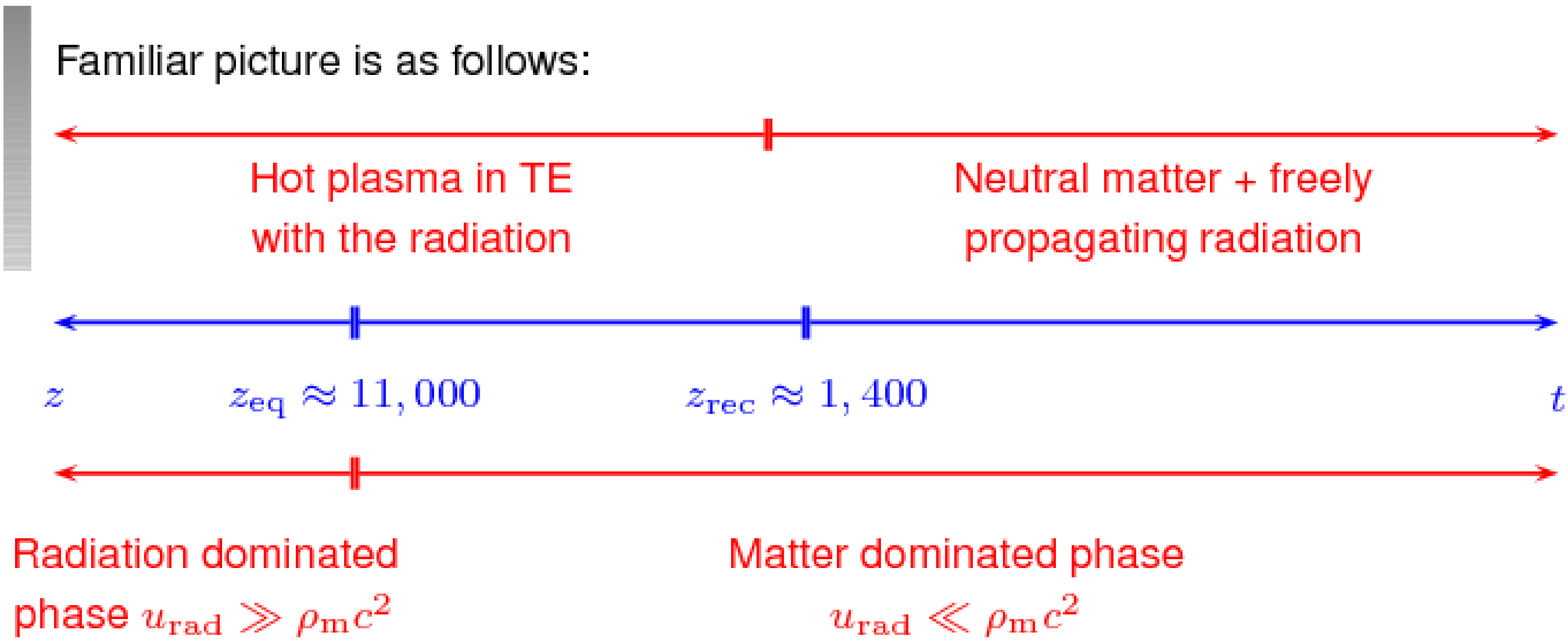
THERMAL HISTORY OF THE UNIVERSE

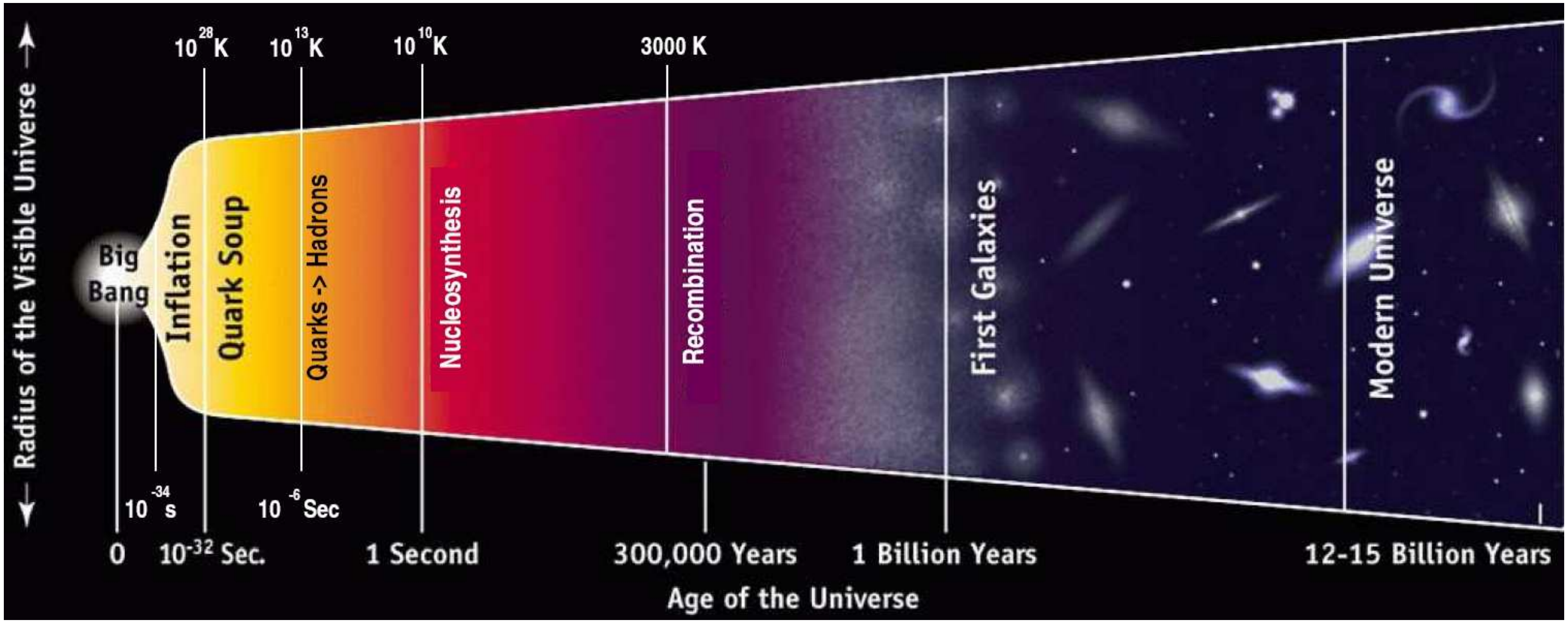
- Now move on to giving a sketch of the important things about 'background universe' evolution we need in order to understand perturbations
- These are basically thermal history, matter versus radiation domination, and horizons
- The universe's energy density is dominated initially by the cosmic microwave background CMB, and this is in thermal equilibrium with matter through to recombination. At this point the universe suddenly becomes transparent, and the photons propagate freely towards us.



EPOCH OF EQUALITY

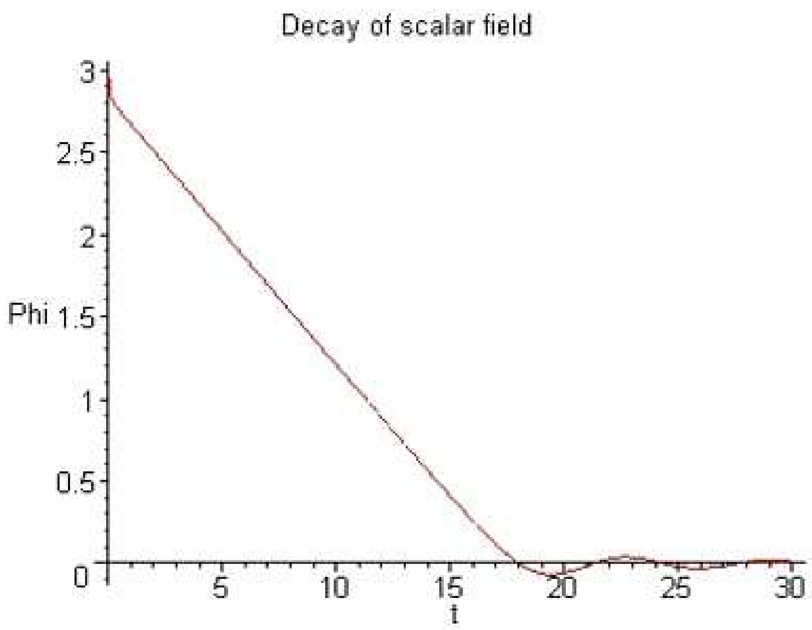
- If scale back from the present day (when matter energy density dominates) to the past, $\rho_{\rm rad}$ increases like $(1+z)^4$ while $\rho_{\rm matter}$ increases like only $(1+z)^3$
- This means there is an epoch when they are equal — timing of this is very important to perturbations, as we'll see
- On all plausible values of parameters, happens before recombination

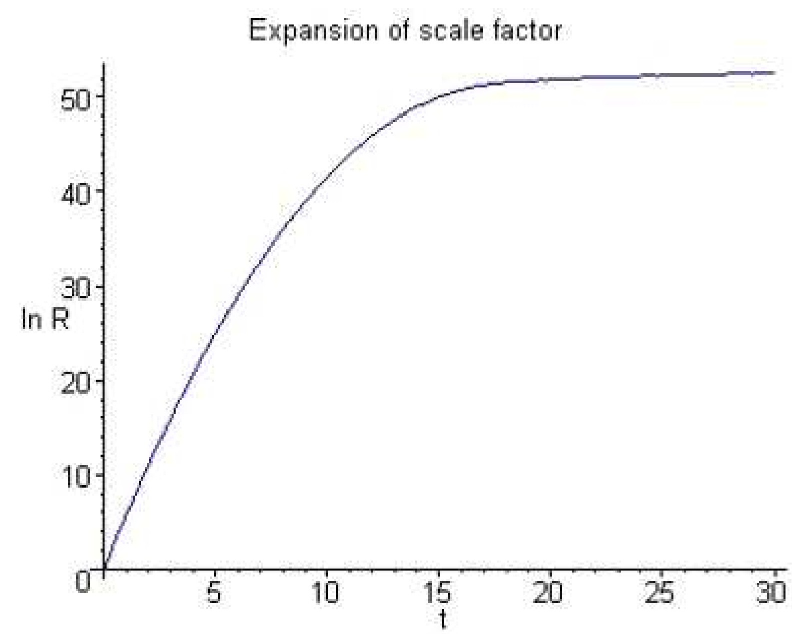




- On this timeline, equality typically happens at about 13,000 years
- Recombination would occur much earlier than at about 4000 K, or 300,000 years, as shown, if it just depended on $13.6\,\mathrm{eV} = kT$
- This in fact would be about 160,000 K!
- Subtleties of recombination are due to two-photon decay process, plus fact that interactions drop out of Saha equilibrium as universe expands
- MSL will cover this

- Remaining background topics concern ‘horizons’ and inflation, plus want to give a preliminary sketch of CMB anisotropies and their description, before we move onto the details
- To understand a bit about horizons and the development of structure, will talk a bit about inflation
- Following two curves show computations for the development of the scalar field and universe scale factor versus time in simple chaotic inflation ($V(\phi) = \frac{1}{2}m\phi^2$)
- The perturbations we are interested in are thought to be generated by quantum field theory fluctuations during the development shown — how do we characterise them statistically?





We are interested in the amplitude of fluctuations as a function of scale. therefore we Fourier analyse them, and express in terms of a *comoving wavenumber* k, related to the physical wavenumber k_p via $k = Rk_p$.

The *power spectrum* is defined as the contribution to the variance of the fluctuations per unit logarithmic interval in k. So if the spatial fluctuations in ϕ are $\delta\phi(\mathbf{x})$, we define

$$\langle \delta\phi(\mathbf{x})\delta\phi^*(\mathbf{x}) \rangle = \int_0^\infty P(k)\, d(\ln k)$$

Now the Fourier transform relation (not worrying about the normalisation) is

$$\delta\phi(\mathbf{x}) \propto \int \delta\phi_k \, e^{i\mathbf{k}\cdot\mathbf{x}} \, d^3k$$

so since $d^3k = 4\pi k^2 dk$ and $d(\ln k) = dk/k$, putting the two together we find

$$P(k) \propto k^3 |\delta\phi_k|^2$$

We can estimate the fluctuations $\delta\phi_k$ via the following heuristic argument. (Don't worry about the details!)

If we were working in a simple Minkowski spacetime, the equation of the scalar field is a simple harmonic oscillator-type equation, and for this it is well-known how to quantise. The basic *mode* with wavenumber k_p is

$$\delta\phi_{k_p} \approx \frac{1}{V^{1/2}} \frac{e^{-ik_p t}}{\sqrt{2k_p}}$$

Here V is some normalising volume, and one can get the factors by demanding that the 'norm' of state (basically the integral of the time component of the 4d current) evaluates to 1 in Planck units.

Now in our present case, we take the normalising volume as being defined by the scalefactor R, so $V \propto R^3$. Converting to comoving wavenumber, we thus get

$$\delta\phi_k \propto R^{-3/2} \frac{e^{-ikt/R}}{\sqrt{2k/R}} = \frac{e^{-ikt/R}}{R\sqrt{2k}}$$

Feeding this into the expression for the power spectrum, we get

$$P(k) \propto \left(\frac{k}{R}\right)^2$$

- Now we need to get to grips with a key feature of the development of perturbations
- This is related to the concept of the ‘comoving Hubble radius’
- We can see that c/H ($= 1/H$ here, since we are taking $c = 1$) is a distance
- It is called the *Hubble radius* or *Hubble length*
- We can make it a comoving measure by dividing by R, and the resulting quantity is

$$\boxed{\frac{1}{RH} = \text{comoving Hubble radius}}$$

Note:

WARNING!

This distance is often called the *comoving horizon radius*. However, ‘horizon’ here is being used in a loose sense. The quantity c/H generally does work out to be some multiple of the proper distance across the *particle* horizon.

For example, in Einstein de Sitter case we have $H = 2/(3t)$ and so $c/H = (3/2)ct$, which is half the proper distance across the particle horizon.

However, we should be clear that:

particle horizon exists at time t if $\chi_p(t) = \int_0^t c\,dt'/R(t')$ is finite

proper distance across particle horizon at time t = $R(t)\chi_p(t)$

whilst

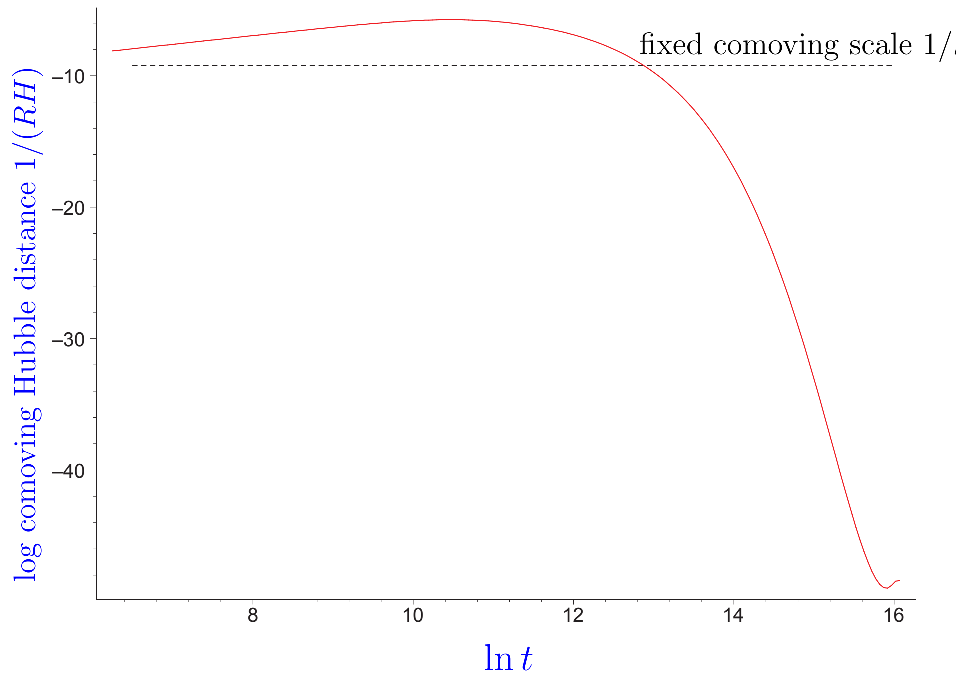
horizon as a word can mean any of ‘particle’ horizon, or ‘Hubble radius’ or the horizon of a black hole (which technically is known as an **event**, rather than particle horizon).

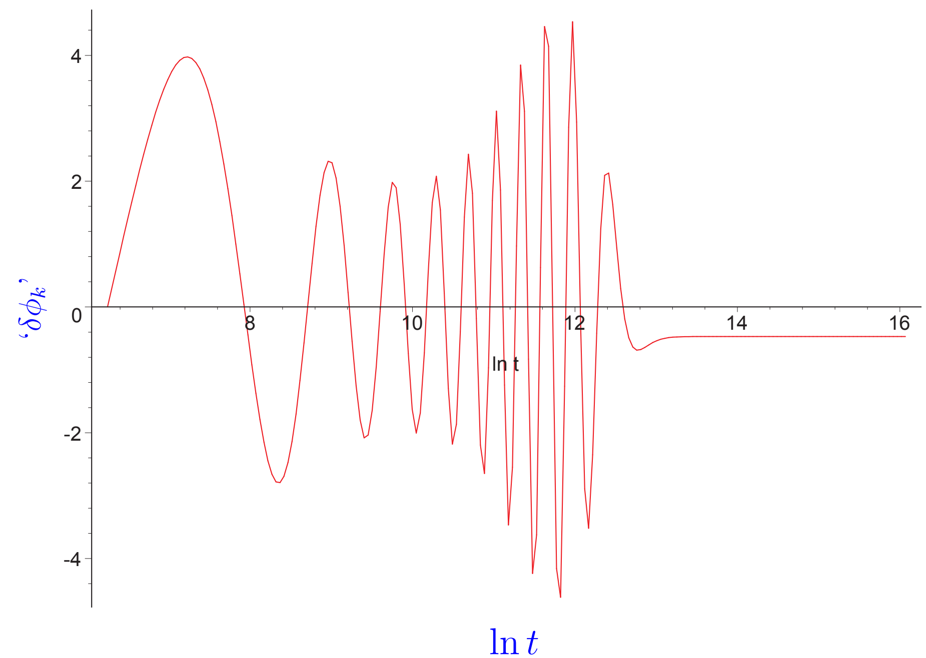
- Carrying on, let us look more closely at the comoving Hubble radius (or ‘comoving horizon’, in the loose way of speaking) in relation to inflation
- H is by definition $\dot{R}/R$ and so RH is in fact just $\dot{R}$.
- We thus see that

inflation is ocurring $\Longleftrightarrow \ddot{R} > 0$
$\Longleftrightarrow$ the comoving horizon distance ($= 1/\dot{R}$) is decreasing

- Thus a fluctuation with a fixed comoving scale $1/k$, can 'start within the horizon' (we think this is how all the quantum perturbations start)
- but then as time goes on during inflation, they will suddenly find themselves 'outside the horizon', since the latter is shrinking. This is illustrated in the next figure.

- How does the fluctuation respond to this?
- During the period it is inside the horizon, it oscillates happily.
- However, as soon as it leaves the horizon (i.e. when $1/(RH)$ becomes less than $1/k$), then suddenly it 'freezes'.
- We can think of this as oscillations needing causal connectedness ('one part needs to feel another part') and once this is lost, it can no longer oscillate, and so stays where it had got to just before horizon crossing.
- This effect is illustrated in the second plot, again from a real computation for the fluctuation evolution (see Hobson, Efstathiou & Lasenby for details)





We can now finally evaluate the power spectrum of the fluctuations, $P(k)$.

The expression we found before

$$P(k) \propto \left(\frac{k}{R}\right)^2$$

clearly needs to be evaluated at 'horizon crossing', since nothing changes after that. At horizon crossing, $k = RH$, so we obtain

$$P(k) \propto H^2\Big|_{\text{horizon crossing}}$$

But during inflation, H is more or less constant.

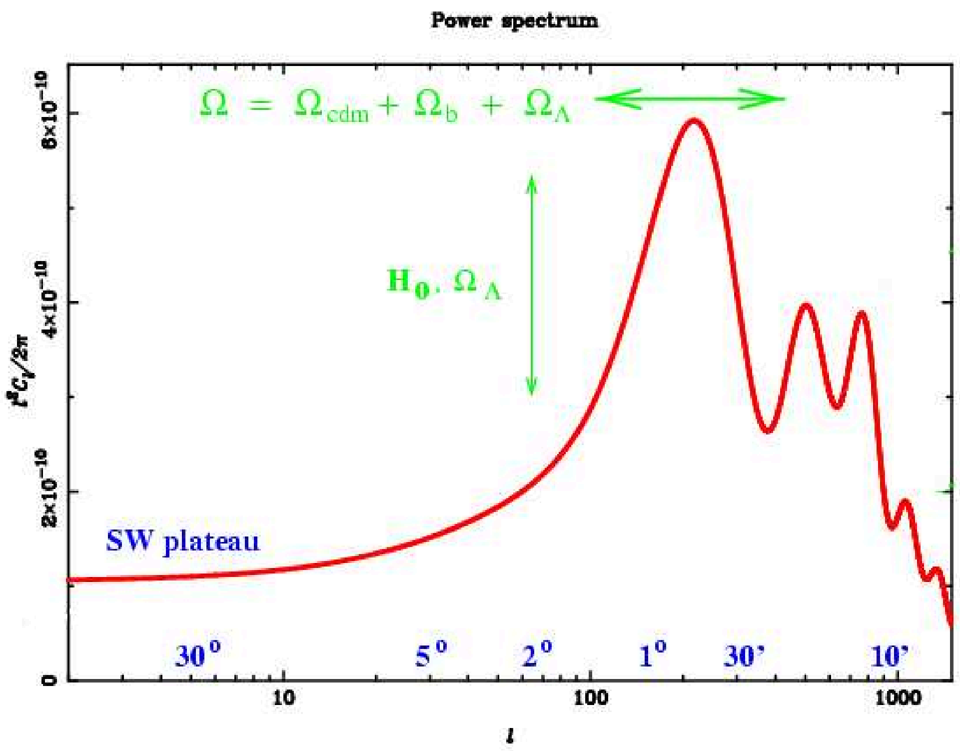
- See for example the plot of $\ln R$ versus t earlier
- Straight line section implies $H \sim \text{const.}$ since H is by definition the slope of $\ln R$ versus t.)
- So $P(k)$ is more or less constant as well.
- I.e., we get approximately equal power per unit logarithmic interval in k as a function of k in the primordial spectrum coming out of inflation!

- This is what provides the primordial spectrum of matter perturbations from which galaxy formation starts.
- So far all evidence points to an approximately scale-invariant spectrum as being in good agreement with the observations.
- We can see effects of it directly in e.g. COBE or WMAP data on large scales (say $\ell \lesssim 50$) (will explain this shortly).
- These have not been processed by 'recombination', and the acoustic oscillations that occur there — and one finds observationally (see next section) that the CMB spectrum there is basically flat.
- In fact might go further, and think about corrections to this prediction.
- If we examine the slope of the $\ln R$ versus t curve shown earlier, we see that H (the slope of the curve), will not be quite constant, but will decrease slightly as time progresses.
- Now, from the plot of $\ln(1/(RH))$ versus $\ln t$ just now, we see that later times correspond to the horizon exit of larger k's.
- Would therefore predict a slightly reduced amplitude of fluctuations at large k (i.e. small linear scales) as compared to small k (large linear scales).

- Amazingly, this prediction (that the n_s in $P(k) \propto k^{n_s - 1}$) is just less than 1, which depends on the details of the mechanism of inflation, is also borne out by the observations, and we are beginning, from the CMB, to get hard quantitative evidence about the dynamics of inflation.

IMPRINTS ON CMB — THE POWER SPECTRUM

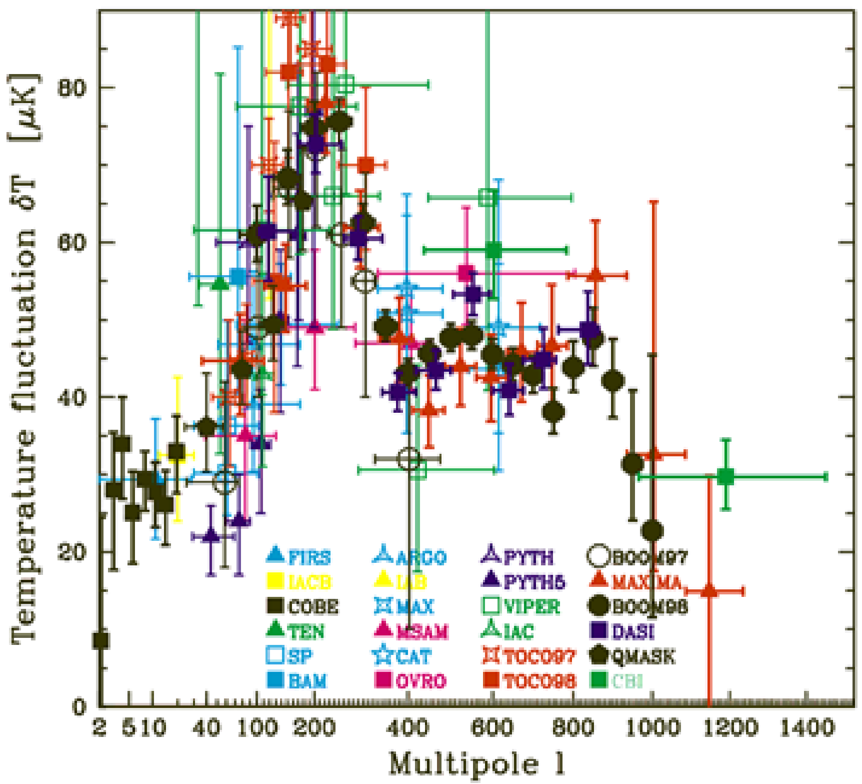
Figure shows the power spectrum as predicted in a particular theory. It is expressed in terms of the power at different spherical harmonics, i.e. the sky is decomposed into a set of spherical harmonics $Y_{\ell m}$ and the power in these plotted as a function of ℓ



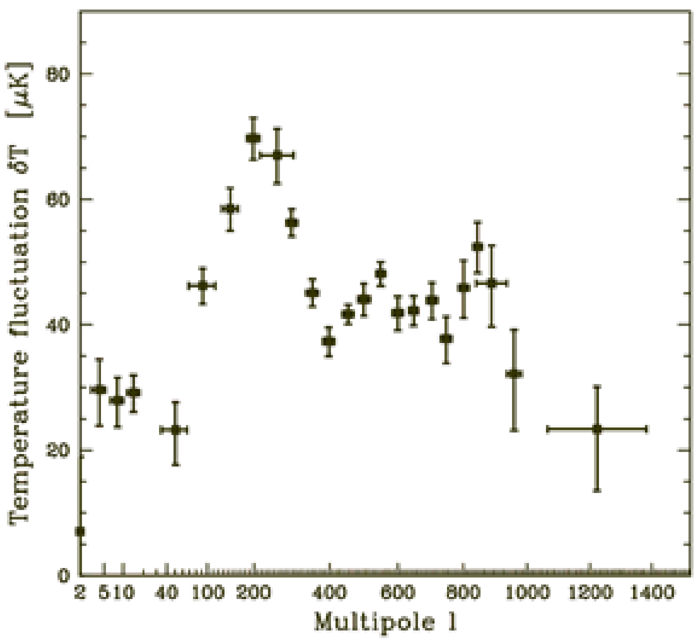
- E.g. $\ell = 1$ is the dipole, $\ell = 2$ is a quadrupole, etc..
- Rough equivalence in terms of angular scale on the sky is shown at the bottom.

- (Note statistical isotropy guarantees that the power is a function of ℓ only, not m.)
- A key predicted feature is the series of peaks at the right of this picture, starting at $\ell \sim 200$, or angular scale $\sim 1^\circ$.
- These are so called 'acoustic peaks', and correspond to coupled acoustic oscillations of the coupled photon/matter fluid during recombination, and have a wealth of physical information encoded in them.
- The flatter region at the left, corresponding to where the COBE measurements were made, is called the Sachs-Wolfe plateau, and helps fix the overall normalisation.
- This is also the region where the reprocessing effects on the primordial spectrum which turn it into the CMB power spectrum during recombination, are minimal, and where we get a direct glimpse of wholly primordial processes.

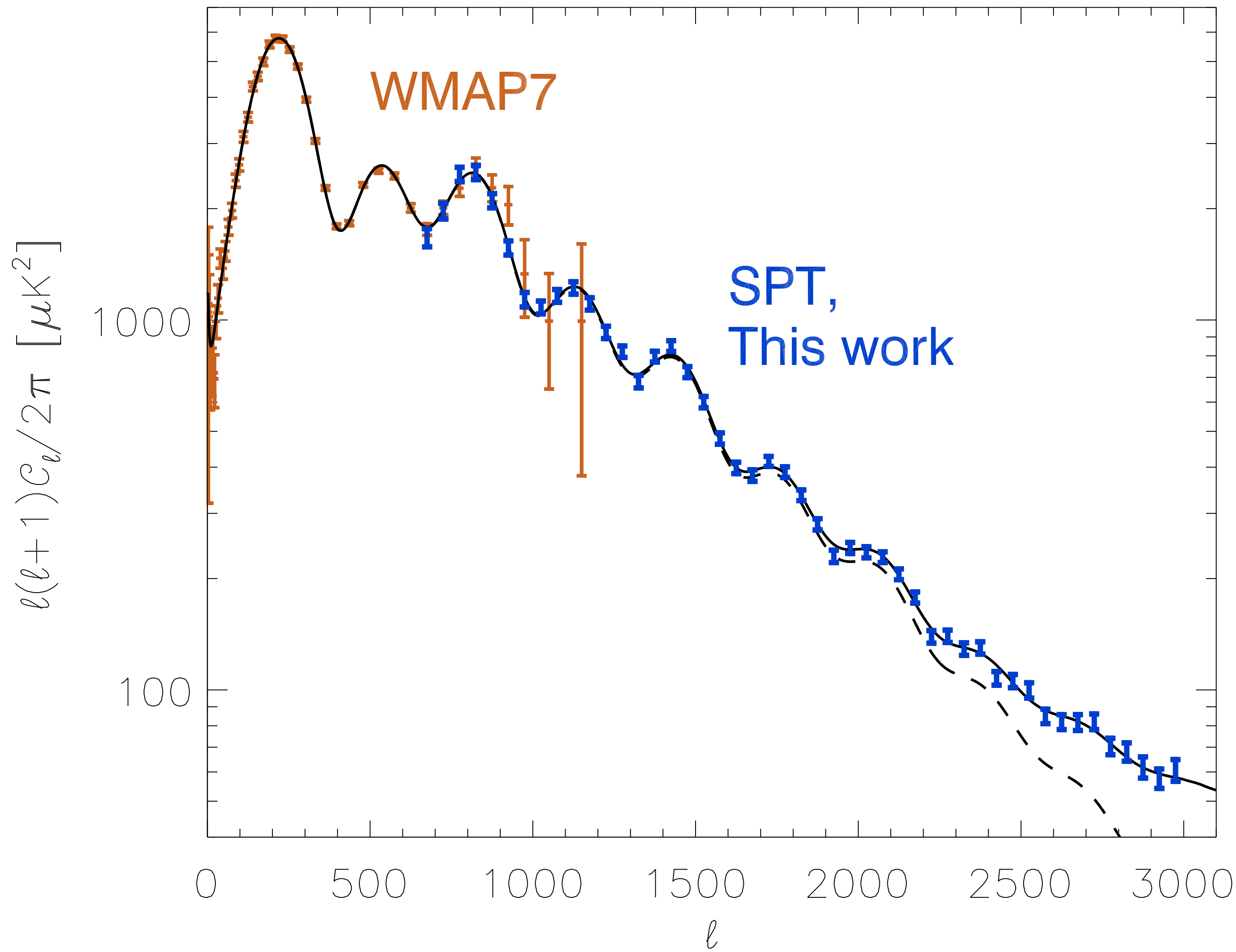
Summary of data May 2001



105 experiments are compressed to 24 bandpowers and covariances – from Wang et al.



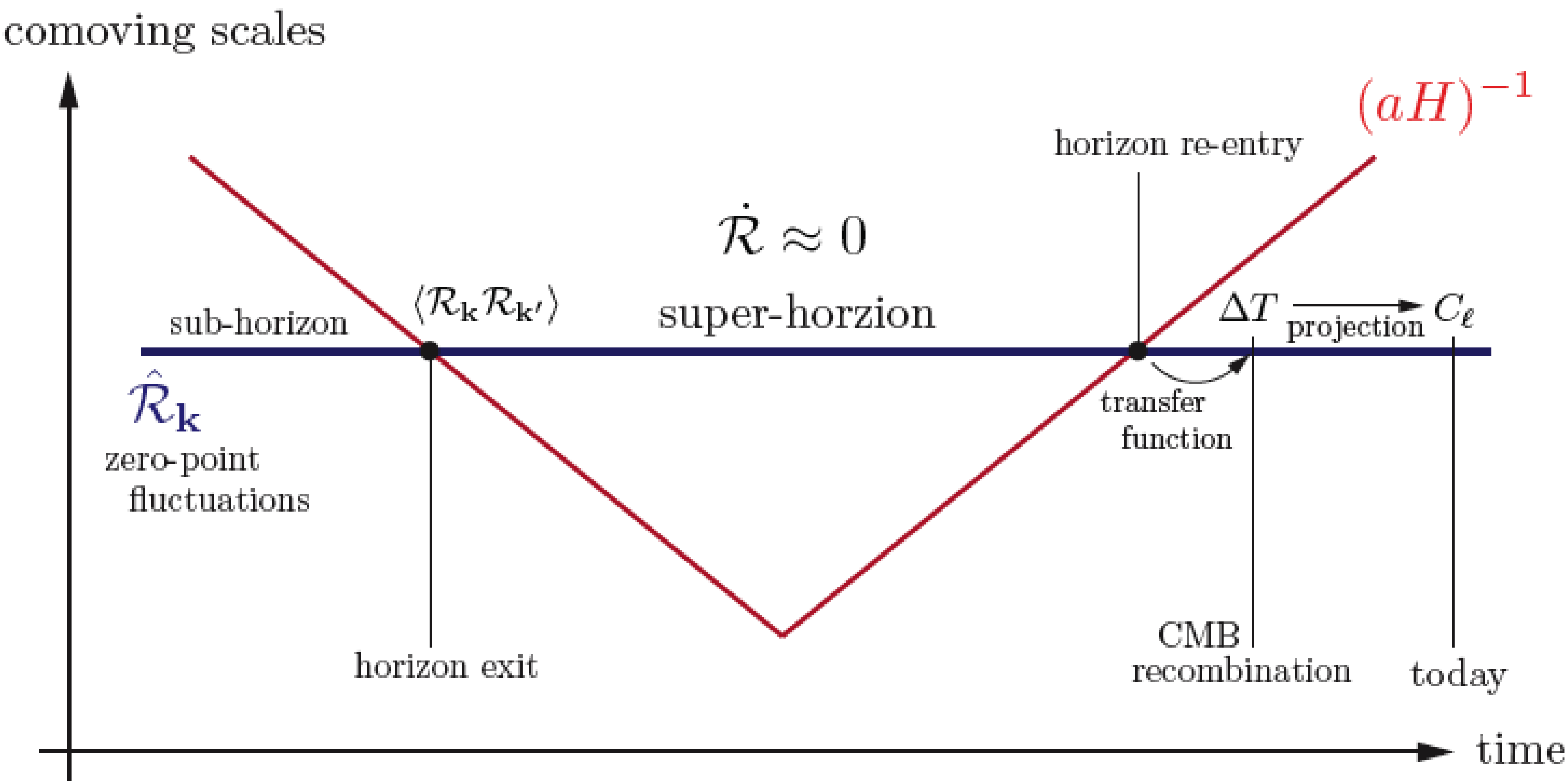
Current WMAP and SPT Power Spectrum Results



Keisler et al (arXiv:1105.318)

- Think one could now reasonably claim that 9 peaks have been measured in the CMB power spectrum!

SUMMARY IN TERMS OF SCALES



(From lecture notes by George Efstathiou)

COSMOLOGY IN CONFORMAL TIME

- Turns out to be convenient when dealing with perturbations to work not with cosmic time, t (which corresponds to the proper time of a comoving observer), but with what's called conformal time, η

- Starting with the usual flat metric

$$ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j$$

with $a(t)$ replacing the previous $R(t)$ as scale factor(!), we transfer the a outside

$$ds^2 = a^2[d\eta^2 - \delta_{ij}dx^i dx^j]$$

defining thereby a conformal time η

$$\eta = \int_0^t \frac{dt}{a(t)}$$

- Since we've ended up with a rescaling by a function $a(\eta)$ of the flat space metric, clear why it's called conformal

Cosmology in Conformal Time

- We can do all this as well for curved models, and for models with a cosmological constant, but actually for our purposes, where we're working mainly with the early universe (up to recombination say), effects of both these are fairly small
- So actually, only explicit cases we're going to need are for the simple flat Einstein de Sitter models, for radiation and matter
- Radiation dominance: Here we know $a(t) = At^{1/2}$, for some constant A
- Using this, we can work out the results

$$\eta = \frac{2}{A} t^{1/2}, \quad a(\eta) = \frac{A^2}{2}\eta, \quad H \equiv \frac{1}{a^2}\frac{da(\eta)}{d\eta} = \frac{2}{A^2\eta^2}$$

and these give the crucial result for the comoving horizon distance in radiation domination

$$\frac{1}{aH} = \left(\frac{A^2}{2}\eta\frac{2}{A^2\eta^2}\right)^{-1} = \eta$$

- or

$$(aH)_{\text{rad dom}} = \frac{1}{\eta}$$

- Note how the constant A drops out of this result
- We can also do the same for matter domination
- Matter dominance: Here we know $a(t) = At^{2/3}$, for some constant A
- Using this, we can work out the results

$$\eta = \frac{3}{A}t^{1/3}, \quad a(\eta) = \frac{A^3}{9}\eta^2, \quad H \equiv \frac{1}{a^2}\frac{da(\eta)}{d\eta} = \frac{18}{A^3\eta^3}$$

and these give the crucial result for the comoving horizon distance in matter domination

$$\frac{1}{aH} = \left(\frac{A^3}{9}\eta^2\frac{18}{A^3\eta^3}\right)^{-1} = \frac{\eta}{2}, \quad \text{so this gives} \quad (aH)_{\text{mat dom}} = \frac{2}{\eta}$$

PLAN FOR REST OF LECTURES

- Have now got the background in place for what we need
- Going to begin, as regards perturbations and their effects on the CMB, with something which normally comes near the end — the effects tensor perturbations, ie. gravity waves!
- Only doing this for the effect on temperature (will leave polarization to Jose Alberto), and for these actually this is more of less the simplest perturbation to deal with, hence starting with it!
- Will then lead on to general scalar perturbations expressed within GR — this is core of what we treat, and progress from this through pressureless perturbations (fairly simple) to the coupled photon/baryon fluid (quite complicated)
- Note will not cover all the aspects that are involved in getting through to the final answers for the C_ℓ's
- Have decided this would take about a lecture course with many more lectures to do properly!
- But hope by treating some aspects in detail to give a flavour

- Tensor metric perturbations $ds^2 = a^2[d\eta^2 - (\delta_{ij} + 2h_{ij})dx^i dx^j]$ where h_{ij} is traceless and transverse
- If substitute this into Einstein equations, and use modes of the form $\exp(ik_i x^i)$, then find

$$\ddot{h}_{ij} + 2aH\dot{h}_{ij} + k^2 h_{ij} = 8\pi G \Sigma_{ij}^T$$

where Σ_{ij}^T is the transverse and traceless part of the anisotropic stress

- Express h_{ij} in terms of the two independent gravitational wave polarization components, h_+ and $h_\times$
- Can ask how solutions will behave during matter domination
- As we've said, if $a \propto t^{2/3}$ then $\eta \propto t^{1/3}$ and then $a \propto \eta^2$ means that the combination aH is $2/\eta$
- With no further sources, the equation we need to solve is then of the form

$$\ddot{h} + \frac{4}{\eta}\dot{h} + k^2 h = 0$$

- Two independent solutions:

$$h \propto \frac{1}{\eta^3}(\cos(k\eta)k\eta - \sin(k\eta)) \quad \text{and} \quad h \propto \frac{1}{\eta^3}(\cos(k\eta)k\eta + \sin(k\eta))$$

- Find following behavior as one goes back in time: only one of these is non-singular, and here can write

$$h_{+,\times}(\eta) = -\frac{3}{k^3\eta^3}(\cos(k\eta)k\eta - \sin(k\eta))h_{+,\times}(0) = 3\sqrt{\frac{\pi}{2}}\frac{J_{3/2}(k\eta)}{(k\eta)^{3/2}}h_{+,\times}(0)$$

- So this is solution for scales entering the horizon well after matter domination

- Note that overall η dependence for large η is $1/\eta^2 \propto 1/t^{2/3} \propto 1+z$, so the gravitational perturbations redshift away inside the horizon — quite unlike the scalar perturbations

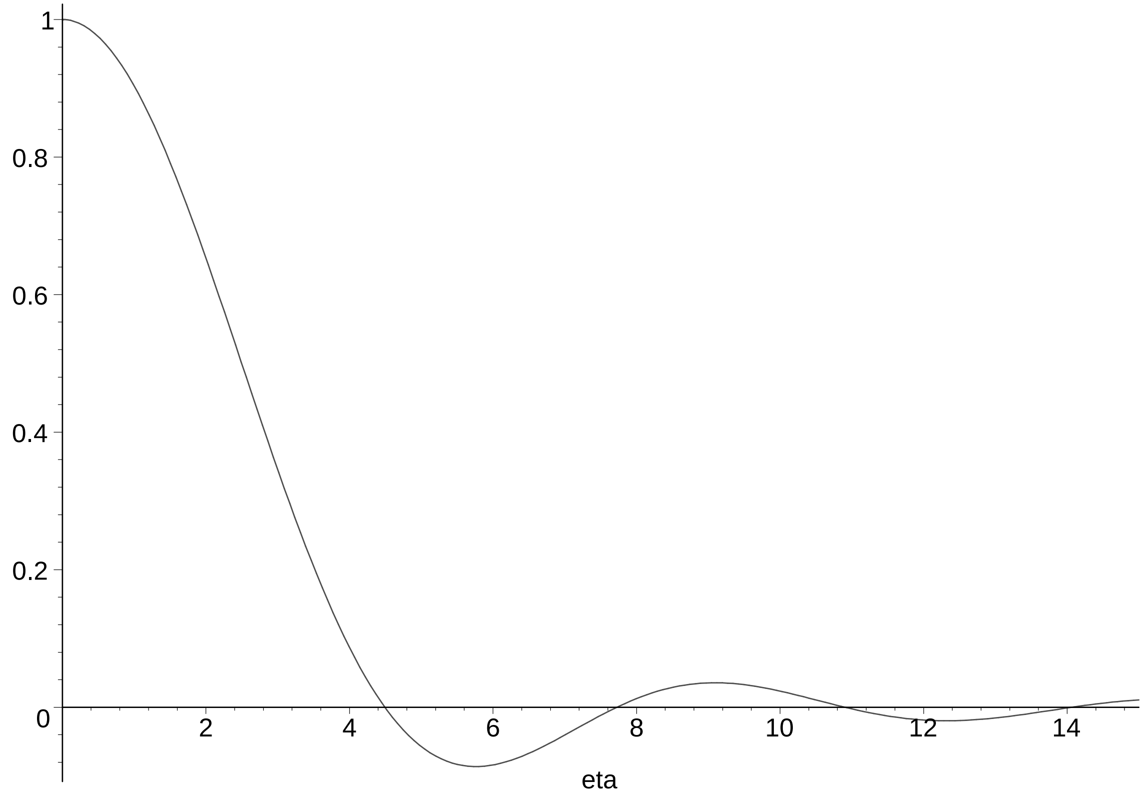
- Therefore only important at degree scales and above

- Can discuss details of the horizon crossing easily in this simple EdS model

- Comoving horizon radius $\frac{1}{RH}$ is $\eta/2$ (found this earlier for matter domination)

- Thus the condition $1/k_{\mathrm{hc}} = 1/(RH)$ yields $k_{\mathrm{hc}}\eta = 2$

- So looking at a plot of $h_{+,\times}(\eta)$ solution we can see if intuition about oscillations starting once horizon has been entered are correct



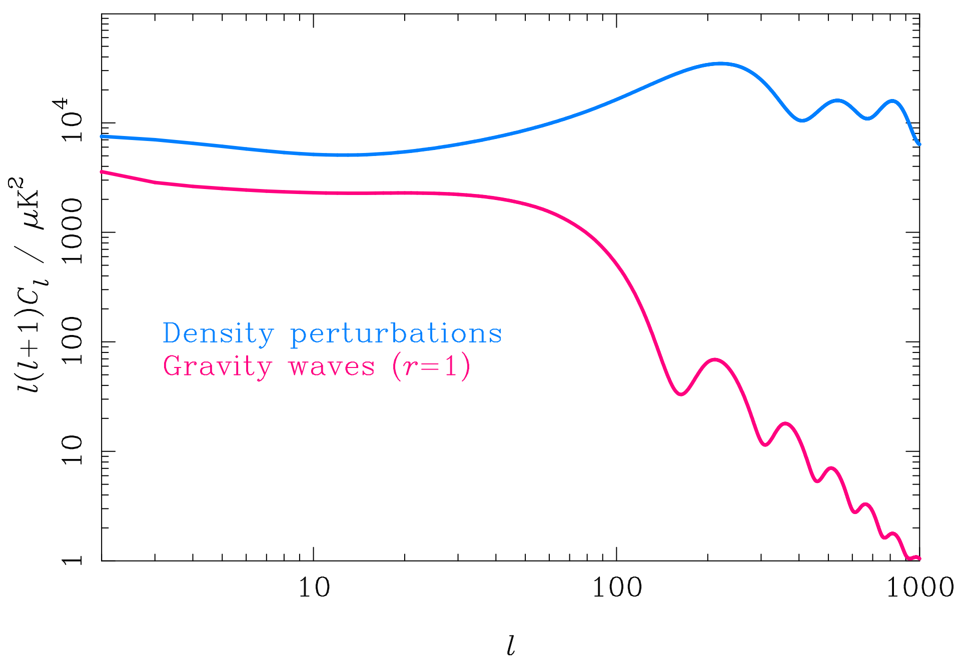
- Indeed see that $k_{\rm hc}\eta = 2$ is a reasonable estimate of where the oscillations begin
- Since have assumed EdS, also easy to work out range of comoving k's under which assumption of using solution corresponding to matter domination is valid (note details here illustrative — real universe of course has Λ, amongst other things)
- In EdS, matter-radiation equality happens at about 13,000 years, and current age

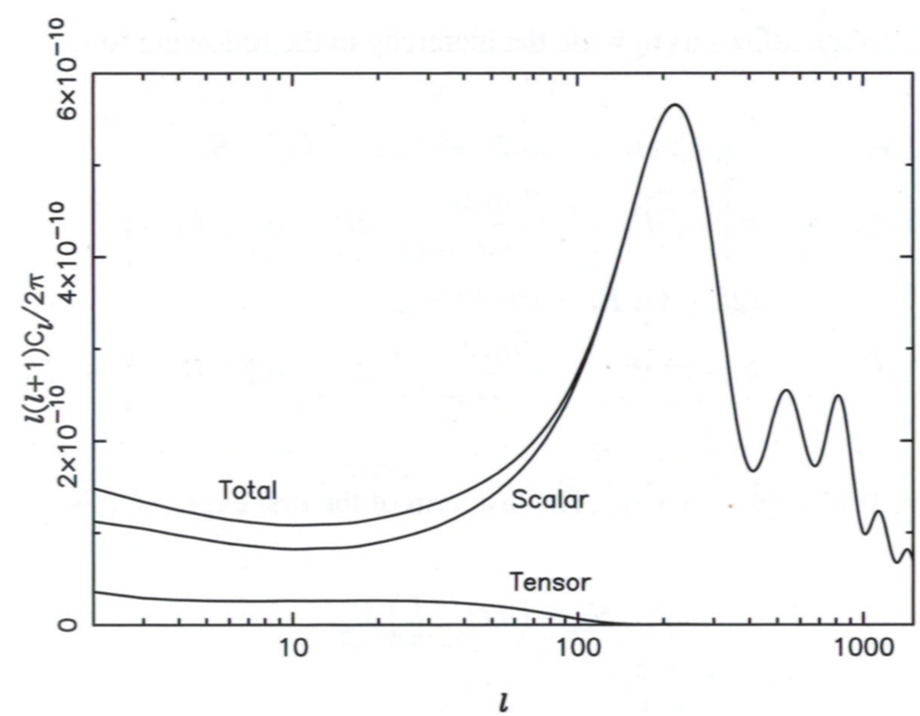
(determined from H_0) would be about 9 Gyr, thus

$$\frac{\eta_{\text{equality}}}{\eta_{\text{now}}} = \left(\frac{13 \times 10^3}{9 \times 10^9}\right)^{1/3} \approx 0.011$$

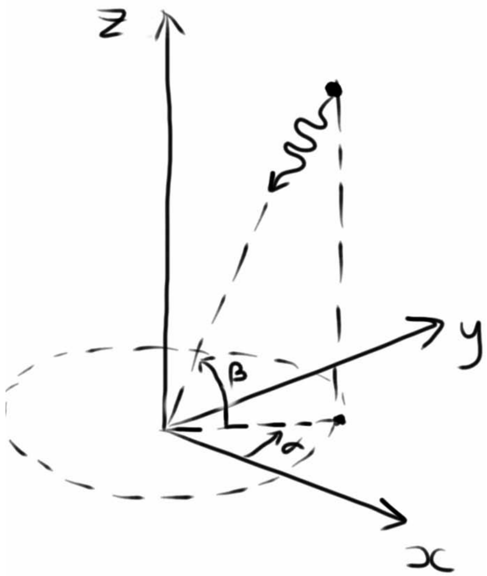
and so scales satisfying our assumption will be up to about 1/100th of the current horizon scale

- Jumping to the answer for currently believed models, we find the following graphs for predicted tensor component in the temperature of CMB, confirming our general picture





- The r here refers to the ratio of the scalar $P(k)$ to the gravitational one at some given k (typically low)



- But how does the h_{ij} perturbation actually get transferred to the CMB temperature?

- Let's be more specific about our gravitational wave

- Take direction of propagation to be z, then can encode transverse, traceless nature via

$$h_{11} = h_+(t,z), \quad h_{12} = h_\times(t,z), \quad h_{22} = -h_{11}, \quad h_{21} = h_{12}$$

and all other h components 0

- Now assume photon motion is inwards, making angle β with plane of polarization of wave, and at an angle α within plane

- If p is the photon 4-momentum, and v the velocity of a comoving observer, then define observed frequency by $h\nu = p \cdot v$

- Then the geodesic equation in this metric leads (to first order) to

$$\dot{\nu} = -\nu\frac{\dot{a}}{a} + \tfrac{1}{2}\nu\cos^2\beta\left(\dot{h}_+\cos 2\alpha - \dot{h}_\times \sin 2\alpha\right)$$

- The first part is just the ordinary redshift, the second gives the perturbation we are after
- Employing our generalized Liouville approach, know that the temperature perturbation must satisfy

$$\frac{\Delta T}{T} = \frac{1}{2}\cos^2\beta\left(\dot{h}_+\cos 2\alpha - \dot{h}_\times \sin 2\alpha\right)$$

- Can clearly see spin-2 nature of interaction of photon with gravitational wave
- Written more generally, and integrating over photon path, we get

$$\Theta(\hat{\boldsymbol{n}}) \approx -\frac{1}{2}\int_{\eta_1}^{\eta_0} d\eta\, \dot{h}_{ij}\hat{n}^i\hat{n}^j$$

  for η_1 some suitable early epoch, which is the basic expression for the gravitational Sachs-Wolfe effect
- As we've seen, only contributes on large scales since h_{ij} decays like a^{-1} after entering horizon

GR TREATMENT OF PERTURBATIONS

- One can give a pretty good account of the development of perturbations of baryons and cold dark matter on sub-horizon scales, as long as the baryons aren't coupled to radiation, using Newtonian perturbation theory
- However, for CMB we do have to consider epoch when photons are baryons are tightly coupled
- Also want to understand effects of horizon scale in a clear fashion, and to be able to deal with epochs when background universe is radiation dominated
- For all this, need to use GR perturbation theory, and of course this includes Newtonian theory as a limit
- So that's what we looking at here, bypassing going through the Newtonian route
- Definitely more complicated, but does all the things we need

GR TREATMENT OF PERTURBATIONS (CONTD.)

- A problem one hits immediately in the GR approach is gauge ambiguities
- Physical results in GR are insensitive to the choice of coordinates (freedom from this is one of the things GR is built upon)
- In cosmology this is particularly acute. E.g. given a scalar perturbation we could define a slicing of spacetime by calling all the points where the scalar has the same value a given slice, and attaching a ‘time’ value to it
- Then the spatial perturbations seem to have gone away, but have they really?
- Two ways round this — either work in gauge-invariant variables (as developed by Bardeen, and see also arXiv:astro-ph/9804301 and arXiv:astro-ph/9804150 by Challinor & Lasenby), or remove the gauge freedom by insisting upon a particular choice of coordinates which achieves this
- Will work via second of these routes, using the Conformal Newtonian Gauge

Reaching the Conformal Newtonian Gauge

- Most general first order metric perturbation is

$$ds^2 = a^2(\eta)\left\{(1-2A)d\eta^2 + 2B_i d\eta dx^i - \left[(1+2D)\delta_{ij} + 2E_{ij}\right]dx^i dx^j\right\}$$

- Here i labels spacelike coordinates, A and D are scalar functions, B_i is known as the shift function (and A the lapse function), and the 3d tensor E_{ij} is distinguished from the δ_{ij} part by being traceless

- This is pretty complicated, and clearly contains all of scalar, vector and tensor perturbations

- Currently, we are only interested in the scalar modes but arbitrary coordinate transformations applied to these can generate all the terms we've got here! (i.e. 'apparent' perturbations)

- But this is actually good! Using such transformations, and knowing that we have physically only scalar perturbations present, then can use such coordinate transformations to 'undo' most of the degrees of freedom in above metric, and find one is left (uniquely) with the following form, where the remaining d.o.f. can't be 'gauged' away

TREATMENT IN CONFORMAL NEWTONIAN GAUGE

- Find one is left with

$$ds^2 = a^2(\eta)\left\{(1-2\Psi)d\eta^2 - (1+2\Phi)\delta_{ij}dx^i dx^j\right\}$$

where we have changed notation with $A \mapsto \Psi$ and $D \mapsto \Phi$

- This is called the Conformal Newtonian Gauge

- 'Conformal' since we use the conformal time η, and Newtonian since we recognise (if we had $\Psi = \Phi$) the approximate linearised form of GR metric in the case where Φ is the Newtonian gravitional potential. (E.g. think of expanding the standard Schwarzschild metric in isotropic coordinates to first order in $\Phi = -GM/r$.)

- Will see shortly that the difference $\Psi - \Phi$ depends on the presence of anisotropic stress

- So will use this metric to calculate the GR connection and curvature tensor, and from this the Einstein tensor $G_{\mu\nu}$, which we will put in the Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

where $T_{\mu\nu}$ is the stress-energy tensor

Treatment in Conformal Newtonian Gauge

- For the latter one needs to adopt a first order perturbed form
- For a fluid, have

$$T^0_0 = \rho + \delta\rho$$
$$T^i_0 = -T^0_i = (\rho + P)v^i$$
$$T^i_j = -(P + \delta P)\delta^i_j - \Sigma^i_j$$

where Σ_{ij} is the anisotropic stress, usually written in dimensionless form as $\Pi_{ij} = \Sigma_{ij}/P$

- Find that the Einstein tensor piece corresponding to this (i.e. removing isotropic component) is

$$\delta G_{ij} = \frac{1}{a^2}\frac{\partial^2}{\partial x^i \partial x^j}(\Psi - \Phi)$$

so for fluids without anisotropic stress (e.g. CDM, or tightly-coupled photon-baryon fluid, but not neutrinos) we can ignore the distinction between the two gravitational potentials

PERTURBATION EQUATIONS

- So one now works out the perturbation equations from the Einstein equations.
- Just two further things we have to be clear on. It is generally much easier for perturbations to work with those corresponding to a given Fourier component. So we will here assume (implicitly) a multiplying functional form of $\exp(i\boldsymbol{k} \cdot \boldsymbol{x})$ for all components
- Note $\boldsymbol{k}$ and $\boldsymbol{x}$ are both comoving quantities here, so we don't need e.g. any extra $a(\eta)$'s in this expression
- Secondly, we are working with the scalar mode for perturbations
- Applied e.g. to velocity, these are those that are derivable by taking the gradient of a velocity potential, and therefore whose curl (the vorticity) vanishes
- This vorticity part is sourced by vector perturbations, and there's two important things to understand about these
  - They always decay with time — i.e. there's no growing mode
  - In standard cosmology with inflation there's no source for them — one needs e.g. active sourcing by topological defects to get appreciable effects from them, so overall we can ignore them here

Perturbation equations

- Since velocity is a gradient, and $\partial/\partial x^i$ brings down an ik_i, we can define a 'scalar velocity potential' V by

$$v_i = \frac{-ik_i}{k} V$$

where we've divided by k so that $V \equiv |\boldsymbol{v}|$

- Note that $\boldsymbol{v}$ is parallel to $\boldsymbol{k}$
- This is the distinguishing feature of scalar velocity perturbations — a vector perturbation is that part of the Fourier transform of the velocity field at a given $\boldsymbol{k}$, which is perpendicular or transverse to $\boldsymbol{k}$, hence also the name 'transverse perturbations' for these
- Two more (much simpler) pieces of nomenclature — call the fractional density perturbation $\delta \equiv \delta\rho/\rho$ and (as before) have $w \equiv P/\rho$
- Now ready to give the full set of perturbation equations
- Lyth & Liddle give the equations in a nice form (which I've checked)

PERTURBATION EQUATIONS

$$\dot{\delta} = -(1+w)(kV - 3\dot{\Phi}) + 3aHw\left(\delta - \frac{\delta P}{P}\right)$$ Continuity equation

$$\dot{V} = -aH(1-3w)V - \frac{\dot{w}}{1+w}V + k\frac{\delta P}{\rho + P} - \frac{2}{3}k\frac{w}{1+w}\Pi = k\Psi$$ Euler equation

$$\delta + 3\frac{aH}{k}(1+w)V = -\frac{2}{3}\left(\frac{k}{aH}\right)^2\Phi$$ 'Poisson equation'

$$\Pi = \left(\frac{k}{aH}\right)^2(\Psi - \Phi)$$ Anisotropic stress equation

- Notice the second two don't involve derivatives — they are constraint equations
- There's also an equation which Lyth & Liddle don't give, but which is equally important in defining a set of propagation equations and constraints
- This is

$$\dot{\Phi} = \frac{3(aH)^2}{2k}V(1+w) - (aH)\Psi$$

PERTURBATION EQUATIONS

- Can use this in the $\dot{\delta}$ equation, to make the r.h.s. of that be composed of undifferentiated quantities
- Then can see that (in the case where the anisotropic stress vanishes), we have 3 propagation equations for 3 quantities (δ, V and Φ) and 1 constraint
- Differentiating the constraint, and substituting the derivatives from the propagation equations, one finds that the result is just the constraint again
- So can indeed propagate the quantities self-consistently, given some initial conditions
- What are these initial conditions?
- Before considering that, let's just look at some overall features of these equations

- The structure and consequences of these are determined by (a) the equation of state w; (b) the comoving wavenumber k; (c) the comoving horizon radius $1/(aH)$ (note a and H always occur in this combination) and (d) the ratio of these last two, which expresses whether a given mode is (a reminder)

Inside the horizon, so $k > aH,$ Outside the horizon, so $k < aH$

- Will get quite different behaviours depending on the values of these quantities
- Other general feature to note is that we've pretended in writing them that there's one fluid
- In fact we have each of CDM, photons, baryons and neutrinos to worry about during the relevant period
- The equations we've written will be valid for the total perturbations, and of course there's always just one Φ and Ψ which respond to the totals formed by the rest
- If a fluid is not busy exchanging energy and momentum with the other fluids, then each separately will satisfy a version of the equations in which δ and V just for that fluid satisfy the equations

- This is true for the CDM all the time, and the baryons after they lose the pressure support of the photons

- Otherwise, we treat the photons and baryons as a single 'tightly coupled' fluid, with mean values for w etc.

- In particular, since Thomson scattering doesn't change the photon and electron energies, the two fluids satisfy separate continuity equations appropriate to their different w's:

$$\dot{\delta}_\gamma = -\frac{4}{3}kV_\gamma + 4\dot{\Phi}, \quad \dot{\delta}_B = -kV_B + 3\dot{\Phi}$$

but a joint Euler equation for the joint velocity $V_\gamma = V_B$:

$$\dot{V}_\gamma = -aH(1-3w_{\text{eff}})V_\gamma - \frac{\dot{w}_{\text{eff}}}{1+w_{\text{eff}}} + \frac{k}{3}\frac{\rho_\gamma}{\rho_B+\rho_\gamma}\delta_\gamma + k\Psi$$

where the effective equation of state is

$$w_{\text{eff}} = \frac{P_\gamma}{(\rho_\gamma + \rho_B)}$$

- Come back to this, in the meantime:

Solution for pressureless case

- We can get an explicit analytic solution if we assume a single pressureless fluid evolving in a matter dominated background
- So this is appropriate to either CDM at any time after $z_{\rm eq}$, or to the baryons above their Jean's scale after decoupling
- With $w = 0$, the simplest form of equations comes from eliminating V using the 'Poisson equation' constraint
- We get

$$\dot{\delta} = \frac{9(aH)^2 - 2k^2}{6aH}\delta + \frac{27(aH)^4 + 9(aH)^2k^2 - 2k^4}{9(aH)^3}\Phi$$

$$\dot{\Phi} = \tfrac{1}{2}aH\delta + \frac{k^2 + 3(aH)^2}{3aH}\Phi$$

- So have two first order coupled equations
- Standard procedure now is eliminate one of the variables, in order to get a second order equation expressed solely in terms of the remaining variable
- In the linear case, we can generally solve this. Let's do this here to illustrate

- Let's use the $\dot{\delta}$ equation to solve for Φ, and then substitute this into $\dot{\Phi}$ equation
- We get

$$\ddot{\delta} + \frac{aH}{27(aH)^4 + 9(aH)^2k^2 - 2k^4}\left\{\left(81(aH)^4 + 18k^2(aH)^2 - 2k^4\right)\dot{\delta} - \frac{3}{2}aHk^2\left(21(aH)^2 - 2k^2\right)\delta\right\} = 0$$

- Looks complicated, so let's do it other way round instead, i.e. use the $\dot{\Phi}$ equation to solve for δ, and then substitute this into $\dot{\delta}$ equation
- Get

$$\ddot{\Phi} + 3aH\dot{\Phi} = 0$$

- Bingo! Working in matter-dominated background case, so $aH = 2/\eta$, and can quickly solve this to get

$$\Phi = -\alpha - \frac{\beta}{\eta^5}, \quad \text{where } \alpha \text{ and } \beta \text{ are constants}$$

- Subs. back, this leads to the relatively simple δ solution

Solution for pressureless case

$$\delta = \alpha\left(2 + \frac{(k\eta)^2}{6}\right) + \beta\left(\frac{k^2}{6\eta^3} - \frac{3}{\eta^5}\right)$$

- So can see the usual behaviour
- Going back to early epochs, if don't want things to blow up then need to discard the decaying solutions
- So we deduce Φ is constant for pressureless case in matter domination
- So after recombination, this is what actual gravitational potentials do (up to point where assuming Einstein de Sitter no longer appropriate — i.e. Φ starts to evolve once late time effects from Λ start kicking in)
- Meanwhile, solution for density contrast

$$\delta = \alpha\left(2 + \frac{(k\eta)^2}{6}\right)$$

also displays expected features

Solution for pressureless case

- On large scales (small k) it is constant, corresponding to the fluctuations being frozen on super-horizon scales, and for small scales $\delta \propto \eta^2 \propto t^{2/3} \propto a$, which is the expected Newtonian result
- Could say transition between two regimes takes place where second term is (say) 1/3 of first
- So $k\eta > 2$ and $\eta = 2/(aH)$ gives $k > aH$ — expected horizon crossing criterion
- How do we fix the constant α?
- This comes from the primordial perturbations on large scales
- We assume these are adiabatic
- Basically says that the density contrast in each species (CDM, photons, baryons, neutrinos, etc.) is a fixed function of the overall, total density perturbation δ
- The fractions work out as

$$\frac{1}{3}\delta_B = \frac{1}{3}\delta_c = \frac{1}{4}\delta_\gamma = \frac{1}{4}\delta_\nu$$

- This is straightforward (just going with relativistic vs. non-relativistic components)
- The subtle bit is how this relates to the curvature perturbations ζ which are produced by inflation
- Lyth & Liddle discuss this in detail (see their Chap. 5) and find that the gauge invariant definition of ζ corresponds to (with our metric)

$$\zeta = -\Phi + \frac{1}{3}\frac{\delta\rho}{\rho + P}$$

where $\delta\rho$ is the total density perturbation

- Won't go through the details, but can then show from the equations we've already got (evaluated in small k case) that on superhorizon scales we get

$$\Phi = \psi = -\frac{3+5w}{5+3w}\zeta, \quad \delta = -2\Phi$$

- ζ itself remains constant on superhorizon scales all the way since it was generated at inflation
- So for scales entering the horizon during radiation/matter domination the appropriate initial conditions are

$$\Phi = -\frac{2}{3}\zeta, \quad \delta = \frac{4}{3}\zeta \quad \text{radiation dom.} \quad \Phi = -\frac{3}{5}\zeta, \quad \delta = \frac{6}{5}\zeta \quad \text{matter dom.}$$

THE PHOTON/BARYON FLUID

- So we have got ourselves initial conditions
- Time now to face up to the bit involving pressure, i.e. the coupled photon/baryon fluid
- This is quite a bit more complicated, but not impossible, and we follow same route as before in terms of getting a single second order equation
- Remind ourselves of the equations for δ_γ:

$$\dot{\delta}_\gamma = -\frac{4}{3} k V_\gamma + 4\dot{\Phi}$$

- and

$$\dot{V}_\gamma = -aH(1 - 3w_{\text{eff}})V_\gamma - \frac{\dot{w}_{\text{eff}}}{1 + w_{\text{eff}}} + \frac{k}{3}\frac{\rho_\gamma}{\rho_B + \rho_\gamma}\delta_\gamma + k\Psi$$

where the effective equation of state is

$$w_{\text{eff}} = \frac{P_\gamma}{(\rho_\gamma + \rho_B)}$$

THE PHOTON/BARYON FLUID

- This time we leave $\dot{\Phi}$ alone, since this belongs to the total fluid, not just radiation, and solve the first (continuity) equation for V_γ, and substitute this into the Euler equation

- We get (8.57 in Liddle & Lyth, with a few misprints corrected)

$$\frac{1}{4}\ddot{\delta} + \frac{1}{4}\frac{\dot{R}}{1+R}\dot{\delta} + \frac{1}{4}k^2 c_s^2 \delta = F(\eta)$$

where the driving term $F(\eta)$ is given by

$$F(\eta) = -\frac{k^2}{3}\Psi + \frac{\dot{R}}{1+R}\dot{\Phi} + \ddot{\Phi}$$

and

$$c_s^2 = \frac{P}{\rho} \approx \frac{P_\gamma}{\rho_\gamma + \rho_B} = \frac{1}{3(1+R(\eta))} = \quad \text{squared sound speed}$$

and $R \equiv 3\rho_B/(4\rho_\gamma)$ is conventional symbol for a quantity that will vary with epoch $\propto a(\eta)$, and therefore $\dot{R} \propto RH$, i.e. can see second terms correspond to Hubble drag

- So have a damped, forced, harmonic oscillator
- The forcing term itself will oscillate, due to the Ψ and Φ potentials responding to the oscillations of the photon/baryon fluid, and we can attempt a WKB solution in terms of the slowly varying frequency $kc_s(\eta)$
- We find

$$\frac{1}{4}\delta_\gamma(\eta) = A(\eta) + B(\eta)\cos(kr_s(\eta)) + C(\eta)\sin(kr_s(\eta))$$

where the coefficients A, B and C are meant to vary only slowly, and $r_s(\eta)$ is called the sound horizon, and is meant to show the distance a disturbance in the fluid could have traveled since some early time:

$$r_s(\eta) = \int_0^\eta c_s(\eta) d\eta$$

- Note the non-oscillating part $A(\eta)$ corresponds to the non-oscillating part of the driving term coming from Ψ, so can write explicitly

$$A(\eta) = -(1 + R(\eta))\Psi$$

where the $\Psi = \Phi$ at the right is evaluated once potentials constant again

- So here comes an important result. We know that our initial condition linking δ and $\Phi = \Psi$ at early times is

$$\delta(0) = -2\Phi(0)$$

and that this remains roughly constant until horizon crossing

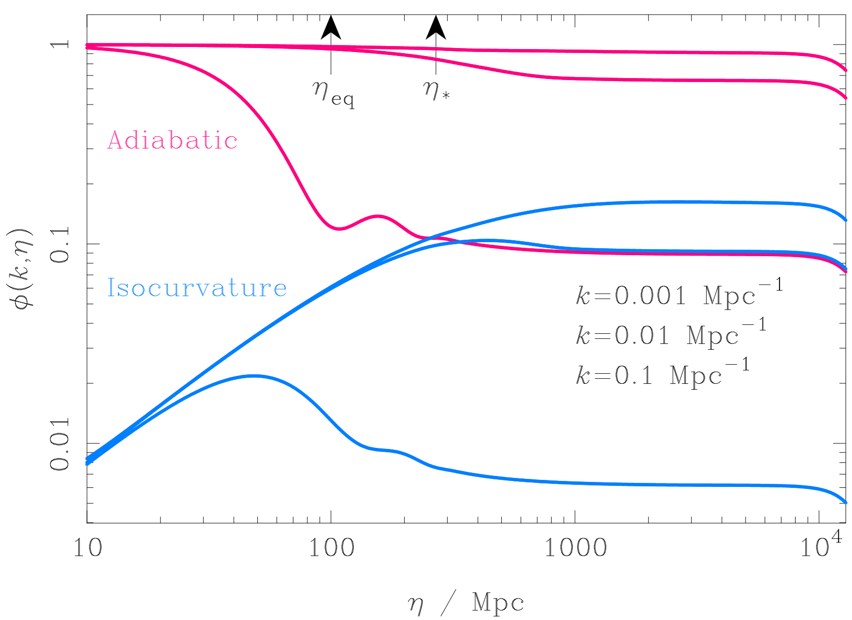
- Meanwhile, WKB method tells us η dependence of the coefficients B and C will be roughly $(1+R)^{-1/4}$, and we know R goes to ~ 0 at earlier times (where radiation dominates)

- Also A term just discussed is in fact quite small for modes that have been oscillating (see below)

- So we see that it is the cosine term that is picked out and excited by the initial conditions

- This is what phases things up

- So we get $B \sim \delta(0) \sim \frac{1}{3}\zeta$, $C = 0$

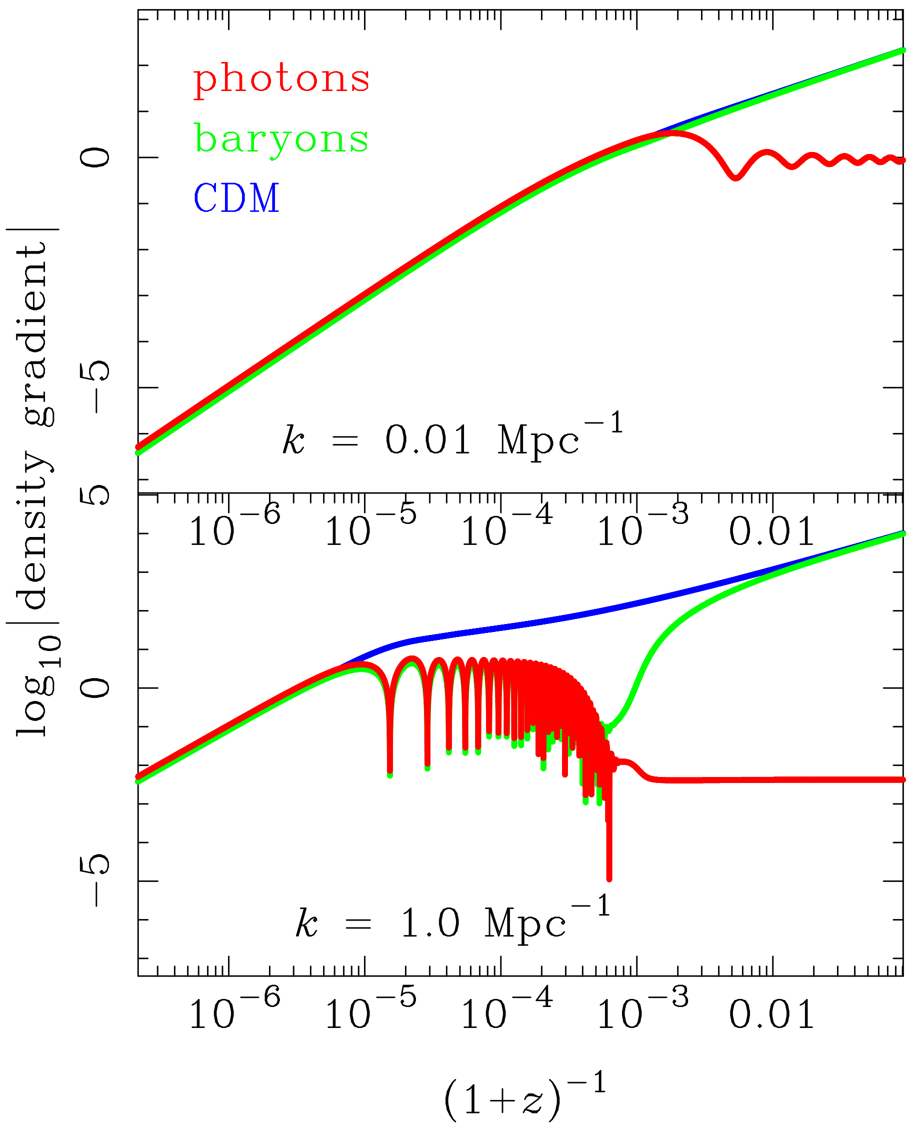


THE RESULTS

- $\delta_\gamma/4$ ($\equiv \Theta_0 = \Delta T/T$) starts out constant at $-\Psi(0)/2 \Rightarrow$ cosine oscillation $\frac{1}{3}\zeta \cos kr_s$ about equilibrium point $-(1+R)\Psi$

- Modes with $k\int_0^{\eta_*} c_s\, d\eta = n\pi$ are at extrema at last scattering $\Rightarrow$ acoustic peaks in power spectrum

- As soon as decoupling of the photon/baryon fluid occurs, baryons fall into the potential wells created by the CDM



What do we see?

- Due to Thomson scattering not changing the electron or photon energy, can use our previous 'Liouville' approach and deduce bb form is maintained, and temperature shifts gravitationally just come from frequency change
- If write $q = ap$, where p is photon momentum, then q is constant in unperturbed universe (this is just standard redshift effect)
- With our Conformal Newtonian Gauge metric, then can show from geodesic equations

$$\frac{1}{q}\frac{dq}{d\eta} = \frac{\partial \Phi}{\partial \eta} + \frac{\partial \Psi}{\partial \eta} - \frac{d\Psi}{d\eta}$$

  where $d\Psi/d\eta$ means a derivative along the photon path

- Then

$$\frac{1}{q}\frac{dq}{d\eta} = \frac{1}{T}\frac{dT}{d\eta}$$

  tells us last term on right gives additional redshift, hence $\Delta T/T$, due to differences in potential Ψ between last scattering point and reception

- Negative contribution to $\Delta T/T$ from potential wells (matter over-densities) at last scattering
- Second term gives *integrated Sachs-Wolfe* contribution

$$(\Delta T/T)_{\mathrm{ISW}} = \int (\dot{\Phi} + \dot{\Psi})\, d\eta$$

ANISOTROPY GENERATION: SCATTERING

- Thomson scattering ($k_B T \ll m_e c^2$) around recombination and reionization dominant scattering mechanism to affect CMB:

$$\frac{d\Theta}{d\eta} = \underbrace{-a n_e \sigma_T \Theta}_{\text{out-scattering}} \underbrace{+\frac{3 a n_e \sigma_T}{16\pi} \int d\widehat{\boldsymbol{m}}\, \Theta(\epsilon, \widehat{\boldsymbol{m}})[1 + (\boldsymbol{e}\cdot\widehat{\boldsymbol{m}})^2]}_{\text{in-scattering}} \underbrace{+ a n_e \sigma_T \boldsymbol{e}\cdot\boldsymbol{v}_b}_{\text{Doppler}}$$

- Neglecting anisotropic nature of Thomson scattering,

$$\frac{d\Theta}{d\eta} \approx -a n_e \sigma_T (\Theta - \Theta_0 - \boldsymbol{e}\cdot\boldsymbol{v}_b)$$

so scattering tends to isotropise in rest-frame of electrons: $\Theta \to \Theta_0 + \boldsymbol{e}\cdot\boldsymbol{v}_b$

- Doppler effect arises from electron bulk velocity $\boldsymbol{v}_b$
  - Enhances $\Delta T/T$ for $\boldsymbol{v}_b$ towards observer
  - Linear effect only important from recombination; non-linear effects from reionization avoid peak-trough cancellation

TEMPERATURE ANISOTROPIES

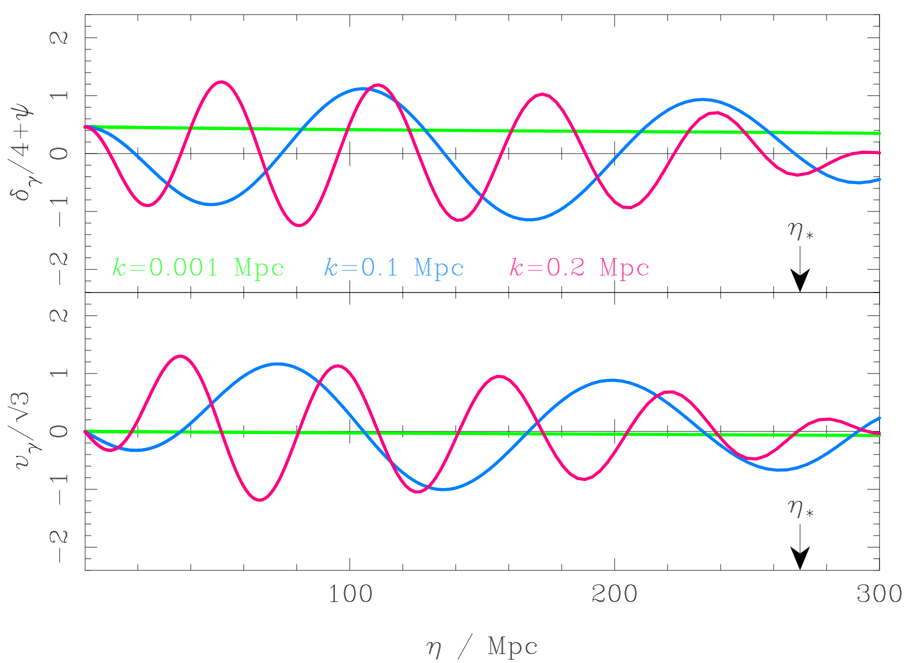
- On degree scales, scattering time short c.f. wavelength of fluctuations and (local!) temperature is uniform plus dipole: $\Theta_0 + \boldsymbol{e} \cdot \boldsymbol{v}_b$
- Observed temperature anisotropy is snapshot of this at last scattering but modified by gravity:

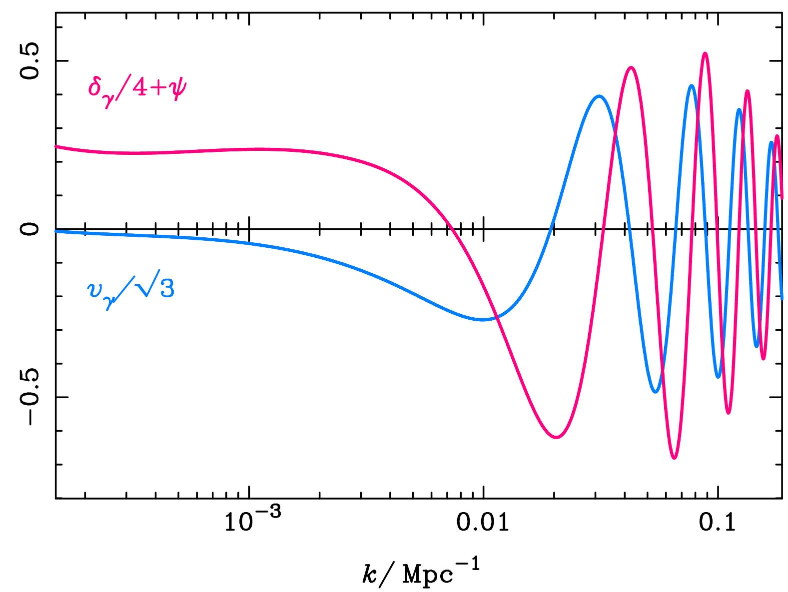
$$[\Theta(\hat{\boldsymbol{n}}) + \psi]_R = \underbrace{\Theta_0|_E}_{\text{temp.}} + \underbrace{\psi|_E}_{\text{gravity}} + \underbrace{\boldsymbol{e} \cdot \boldsymbol{v}_b|_E}_{\text{Doppler}} + \underbrace{\int_E^R (\dot{\psi} + \dot{\phi})\, d\eta}_{\text{ISW}}$$

with line of sight $\hat{\boldsymbol{n}} = -\boldsymbol{e}$, and Θ_0 isotropic part of Θ

  - Ignores anisotropic scattering, finite width of visibility function (i.e. last-scattering surface) and reionization
    - * Will fix these omissions shortly

- So (ignoring ISW part for the moment) sensible to look at time development of separate contributions from $\frac{\delta}{4} + \Psi$ and velocity
  - As we've said, modes with $k \int_0^{\eta_*} c_s \, d\eta = n\pi$ are at extrema at last scattering $\Rightarrow$ acoustic peaks in power spectrum
  - $V_b \approx V_\gamma$ is $\pi/2$ out of phase with Θ_0, as follows from continuity equation (the i that entered in Fourier description) so Doppler effect 'fills in' zeroes of $\Theta_0 + \psi$ and can predict 'Doppler peaks' will be $\pi/2$ out of phase



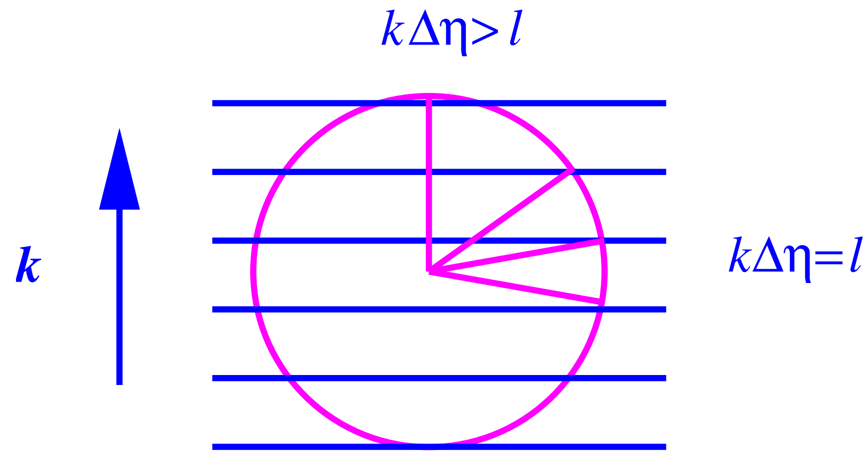


SPATIAL-TO-ANGULAR PROJECTION

- Consider angular projection at origin of potential $\Psi(\boldsymbol{x}, \eta_*)$ over last-scattering surface; for a single Fourier component

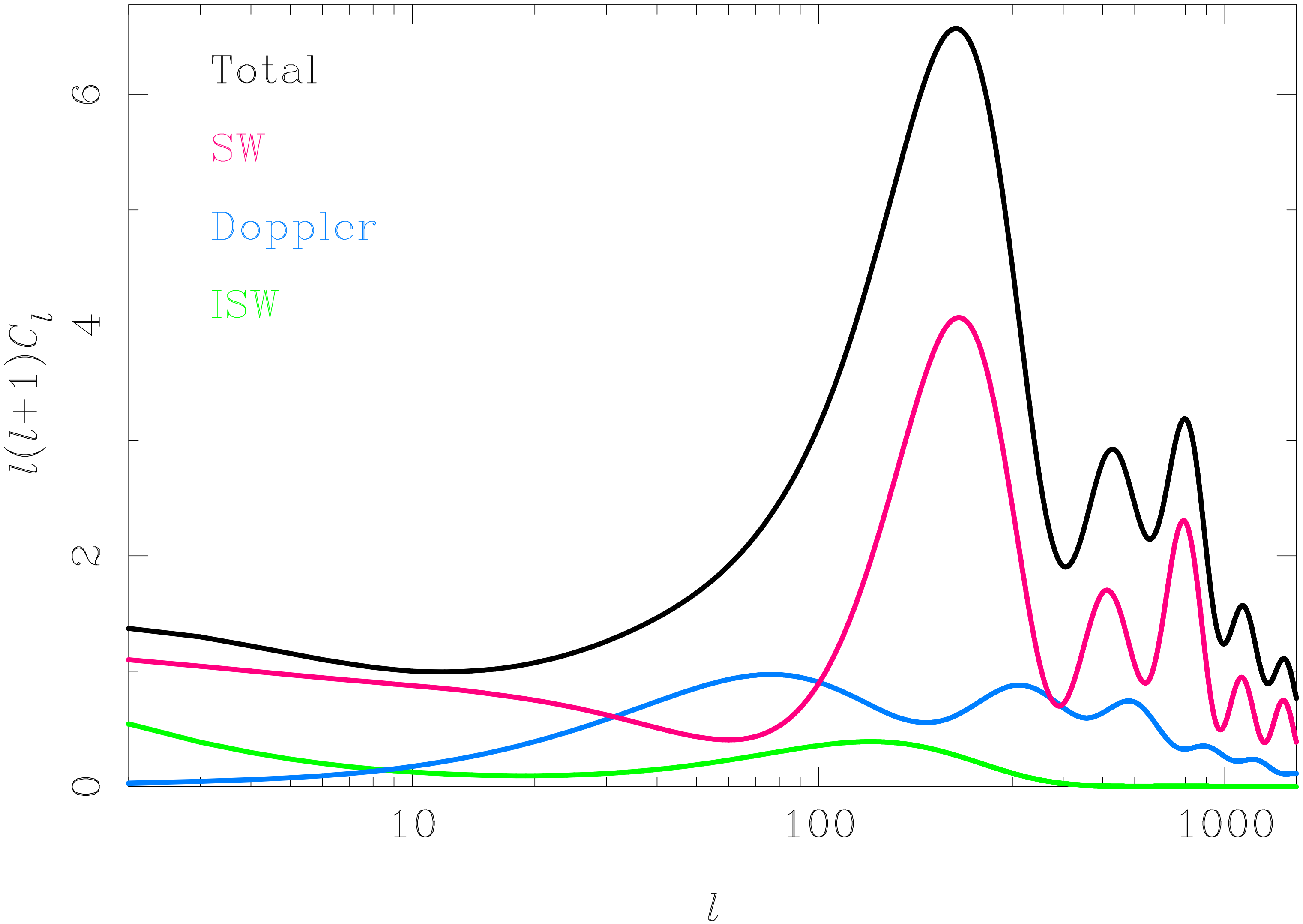
$$\Psi(\hat{\boldsymbol{n}}) = \Psi(\hat{\boldsymbol{n}}\Delta\eta, \eta_*) \qquad \Delta\eta \equiv \eta_0 - \eta_*$$
$$= \Psi(\boldsymbol{k}, \eta_*) \sum_{lm} 4\pi i^l j_l(k\Delta\eta) Y_{lm}(\hat{\boldsymbol{n}}) Y_{lm}^*(\hat{\boldsymbol{k}})$$
$$\Psi_{lm} \sim 4\pi \Psi(\boldsymbol{k}, \eta_*) i^l j_l(k\Delta\eta) Y_{lm}^*(\hat{\boldsymbol{k}})$$

- $j_l(k\Delta\eta)$ peaks when $k\Delta\eta \approx l$ but for given l considerable power from $k > l/\Delta\eta$ also (wavefronts perpendicular to line of sight)



  - CMB anisotropies at multipole l mostly sourced from fluctuations with linear wavenumber $k \sim l/\Delta\eta$ where conformal distance to last scattering $\approx 14\,\mathrm{Gpc}$

- Temperature power spectrum for scale-invariant curvature fluctuations



COMPLICATIONS: PHOTON DIFFUSION

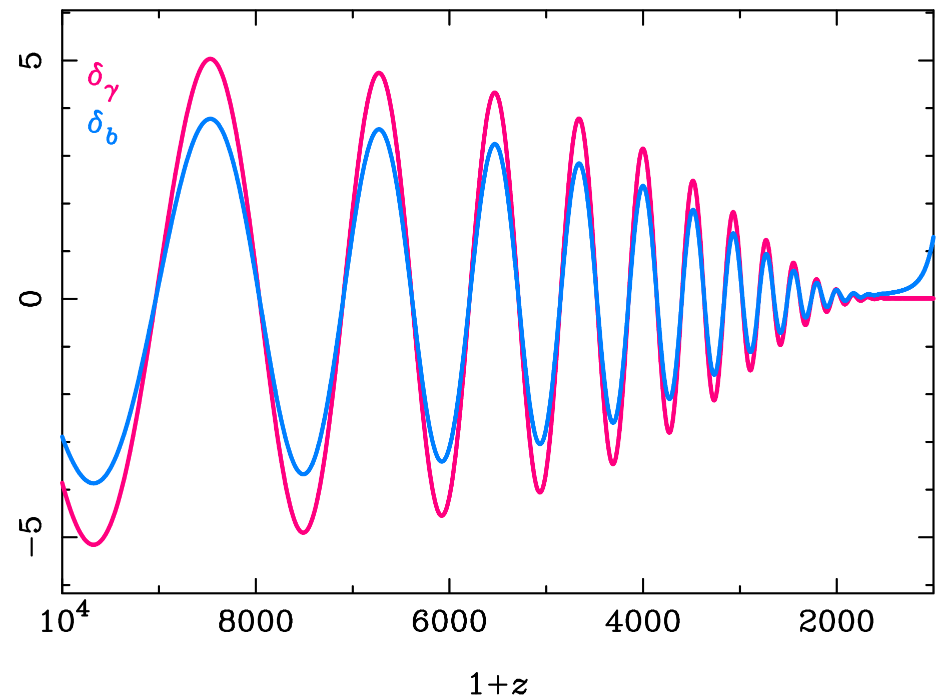
- Photons diffuse out of dense regions damping inhomogeneities in Θ_0 (and creating higher moments of Θ)
  - In time $d\eta$, when mean-free path $\ell = (a n_e \sigma_T)^{-1} = 1/|\dot\tau|$, photon random walks mean square distance $\ell d\eta$
  - Defines a diffusion length by last scattering:

$$k_D^{-2} \sim \int_0^{\eta_*} |\dot\tau|^{-1} d\eta \approx 0.2(\Omega_m h^2)^{-1/2}(\Omega_b h^2)^{-1}(a/a_*)^{5/2}\,\mathrm{Mpc}^2$$

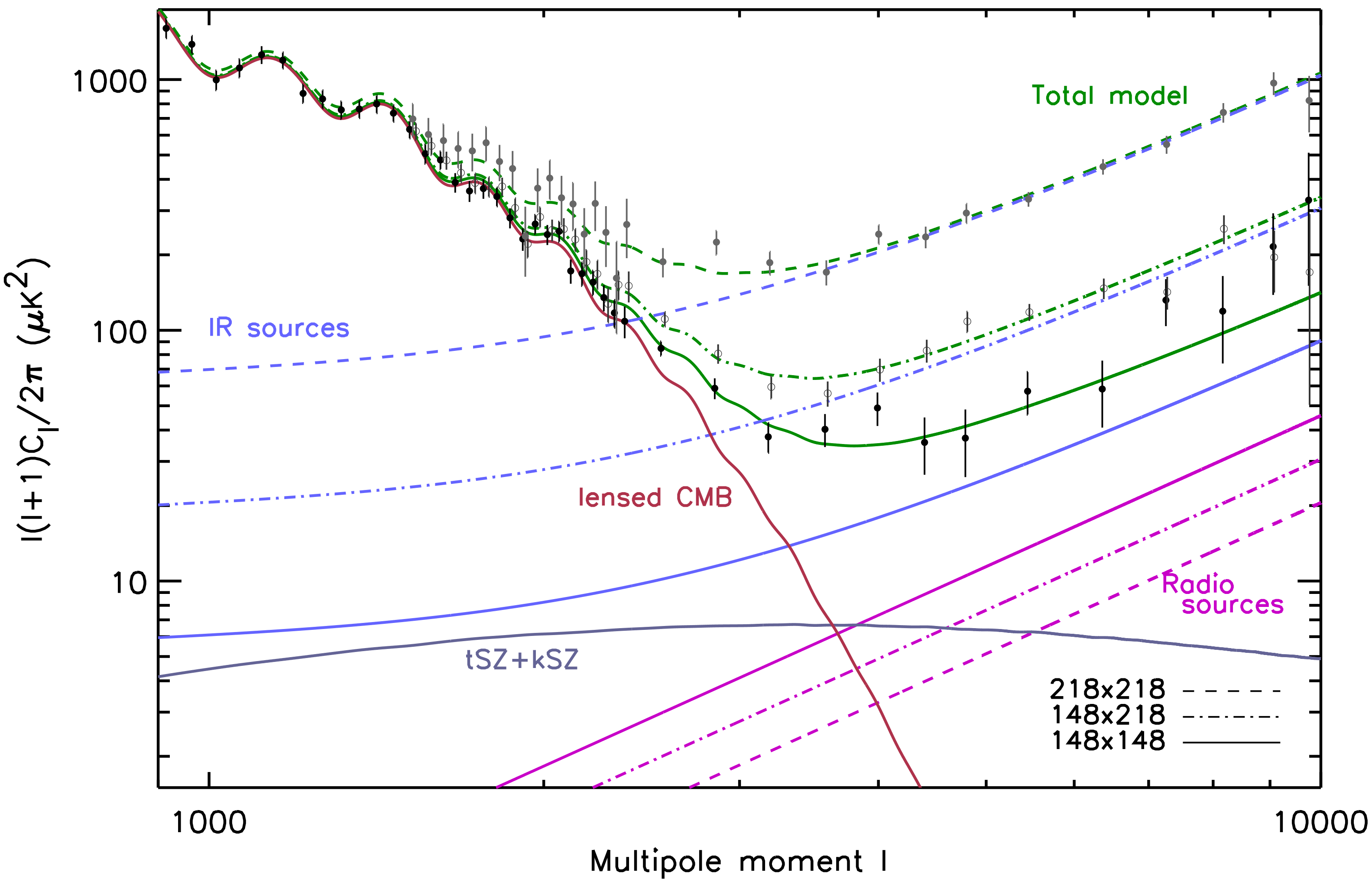
- Get exponential suppression of photons (and baryons)

$$\Theta_0 \propto e^{-k^2/k_D^2} \cos k r_s$$

  on scales below $\sim 30\,\mathrm{Mpc}$ at last scattering
  - Implies e^{-2l^2/l_D^2} *damping tail* in power spectrum
  - Additionally, get extra damping due to finite width $\sigma_z = 80$ of last scattering surface



ACTUAL DAMPING TAIL OBSERVATIONS

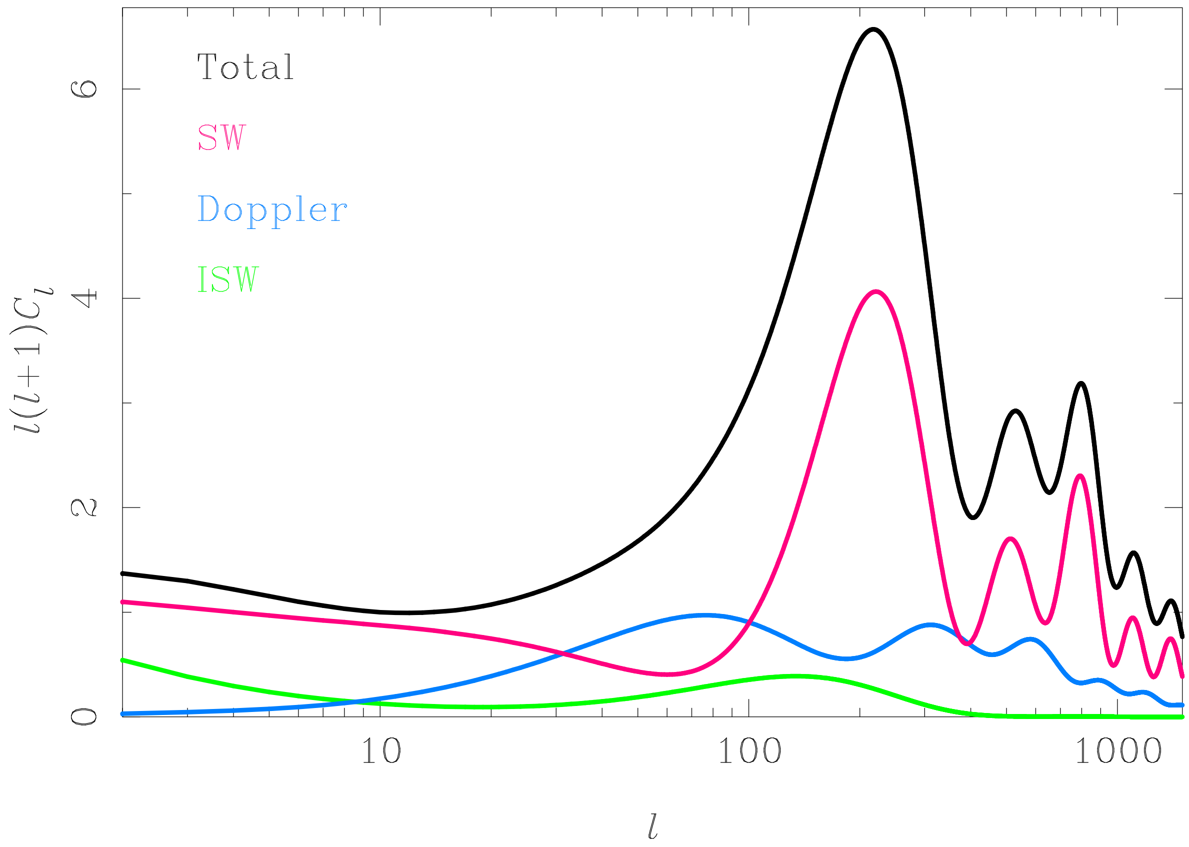


ACT results from Dunkley et al., astro-ph:1009.0866

Notice how SZ effect contributions peak as primordial anisotropies are being strongly damped — mass scale 10^{14} to $10^{15} M_\odot$ of clusters corresponds to width of l.s.s.

INTEGRATED SACHS-WOLFE EFFECT

- Linear $\Theta_{\rm ISW} \equiv \int (\dot\Phi + \dot\Psi)\, d\eta$ from late-time dark-energy domination and residual radiation at η_*; non-linear small-scale effect from collapsing structures
  - In adiabatic models early ISW adds coherently with SW at first peak
  - Late-time effect is large scale (integrated effect $\Rightarrow$ peak–trough cancellation suppresses small scales)
  - Late-time effect in dark-energy models produces positive correlation between large-scale CMB and LSS tracers for $z < 2$

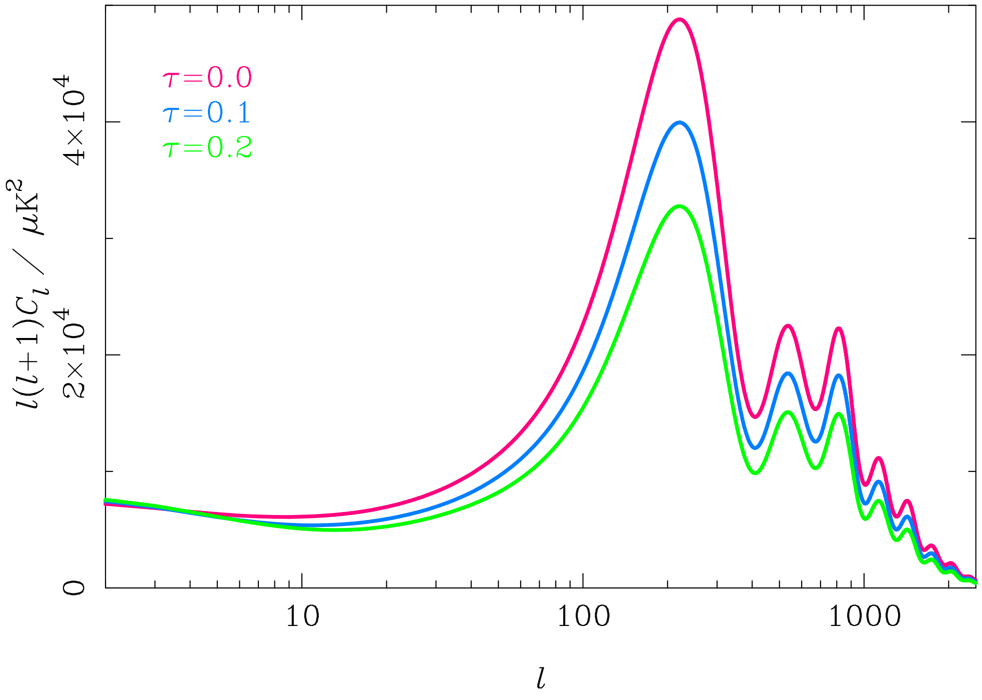


- CMB re-scatters off re-ionized gas; ignoring anisotropic (Doppler and quadrupole) scattering terms, locally at reionization have

$$\Theta(\boldsymbol{e}) + \psi \to e^{-\tau}[\Theta(\boldsymbol{e}) + \psi] + (1 - e^{-\tau})(\Theta_0 + \psi)$$

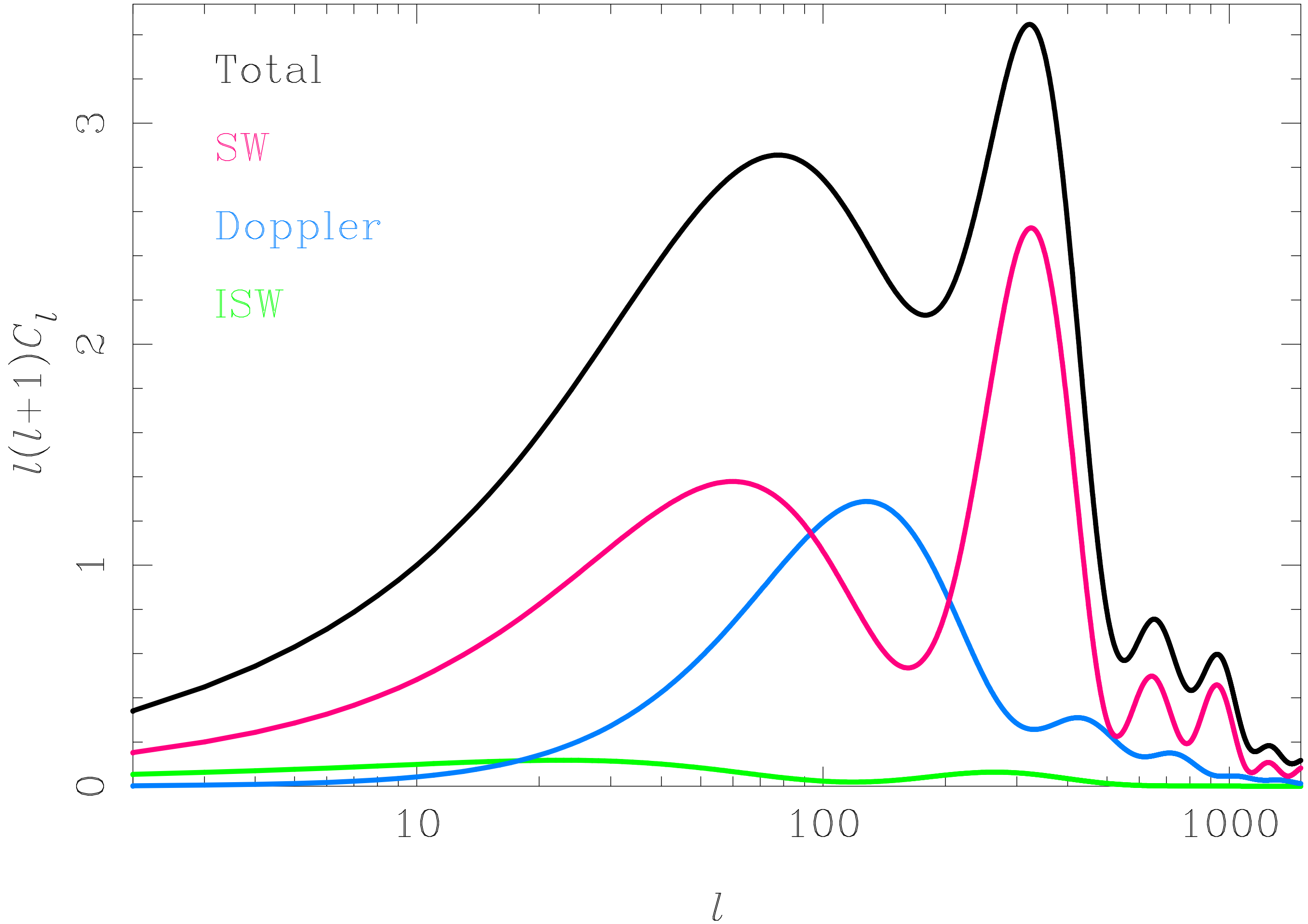
  - Outside horizon at reionization, $\Theta(\boldsymbol{e}) \approx \Theta_0$ and scattering has no effect
  - Well inside horizon, $\Theta_0 + \psi \approx 0$ and observed anisotropies

$$\Theta(\hat{\boldsymbol{n}}) \to e^{-\tau}\Theta(\hat{\boldsymbol{n}}) \quad \Rightarrow \quad C_l \to e^{-2\tau} C_l$$



Isocurvature modes

- So far have said the density contrast in each species (CDM, photons, baryons, neutrinos, etc.) is a fixed function of the overall, total density perturbation δ
- This is the adiabatic condition and the fractions are
$$\frac{1}{3}\delta_B = \frac{1}{3}\delta_c = \frac{1}{4}\delta_\gamma = \frac{1}{4}\delta_\nu$$
- But don't have to assume this. Can decompose into normal modes, and there are 3 extra ones of these where we vary each of δ_c, δ_B and δ_ν separately from the radiation
- CDM isocurvature perhaps most physically motivated (perturb CDM relative to everything else)
- Starts off with $\delta_\gamma(0) = \phi(0) = 0$ so matches onto $\sin kr_s$ modes



Temperature power spectrum for entropy fluctuations (CDM isocurvature mode)

So these can't be dominant, but certainly interesting to set limits on contribution.