Analytical methods for cosmological structure formation: from *N* particles to imperfect fluids

Diego Blas

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CERN

(apologies for the 'guerrilla' style)

Biased Bibliography

(linear part)

Ma and E. Bertschinger '95
Check CLASS webpage (class-code.net) for lecture notes

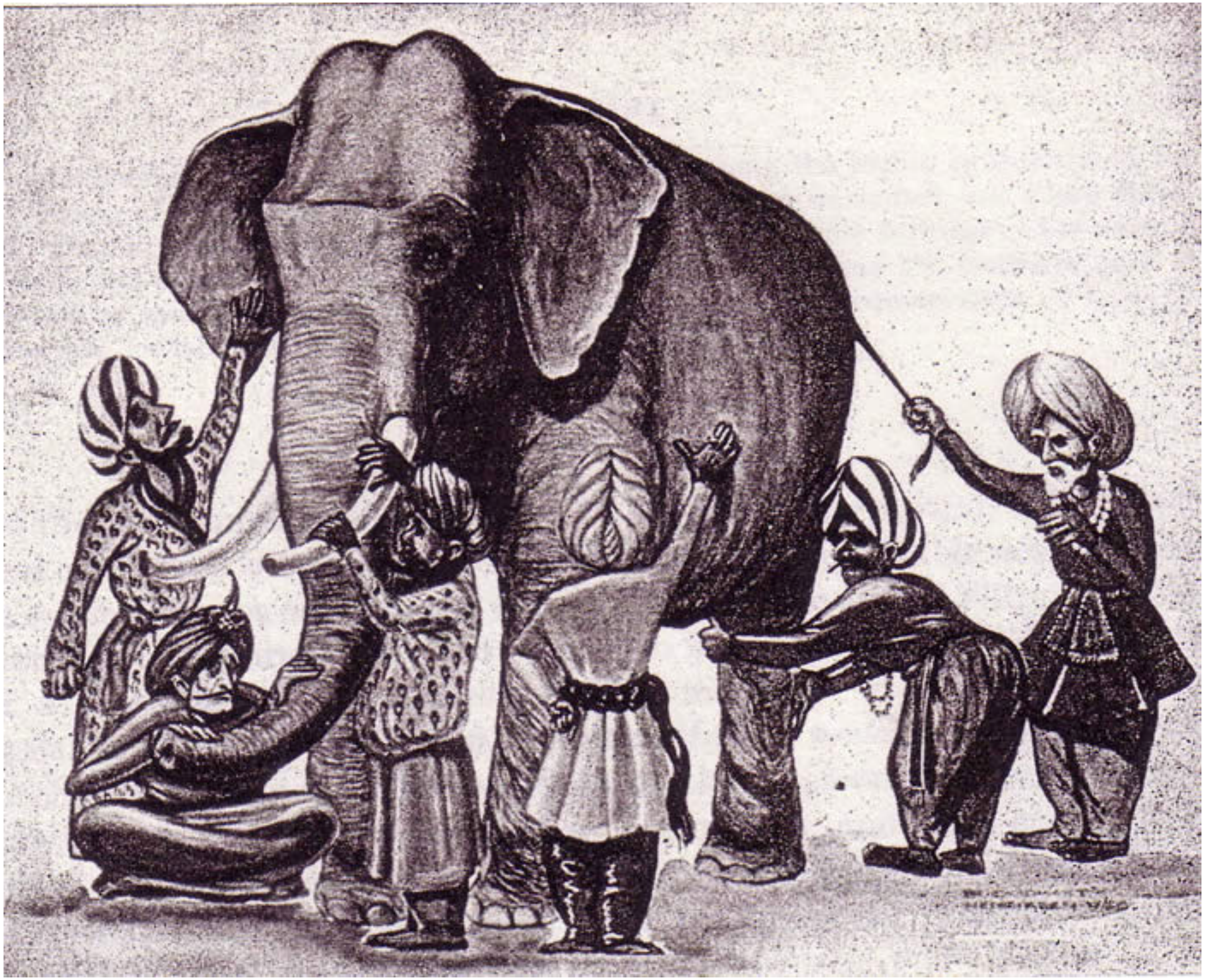
(more on CDM)

Baumann et al. 2010
Zaldarriaga lectures ICTP school 2015
Bernardeau et al. Phys. Rep. 2001

(Couple of books)

Peebles 'The large scale structure of the Universe'
Dodelson 'Modern Cosmology'

Where are we?



Theoretical framework

$$8\pi G T^m_{\mu\nu} = G_{\mu\nu}$$

Matter

Gravity
always small potentials
(relativistic effects at large scales)

i) Background evolution

$$\bar{A} \equiv \langle A_{\mu\nu} \rangle$$ isotropic and homogeneous

$$\rho_n(x,t) = \bar{\rho}_n(t)(1+\delta_n(x,t)) \qquad ds^2 = a(t)^2[-(1+\phi)dt^2 + \delta_{ij}(1+\psi)dx^i dx^j]$$

$$\mathcal{H} = \frac{\dot{a}}{a}$$

$$\mathcal{H}^2 = \frac{8\pi G}{3} a^2 \sum \bar{\rho}_n$$

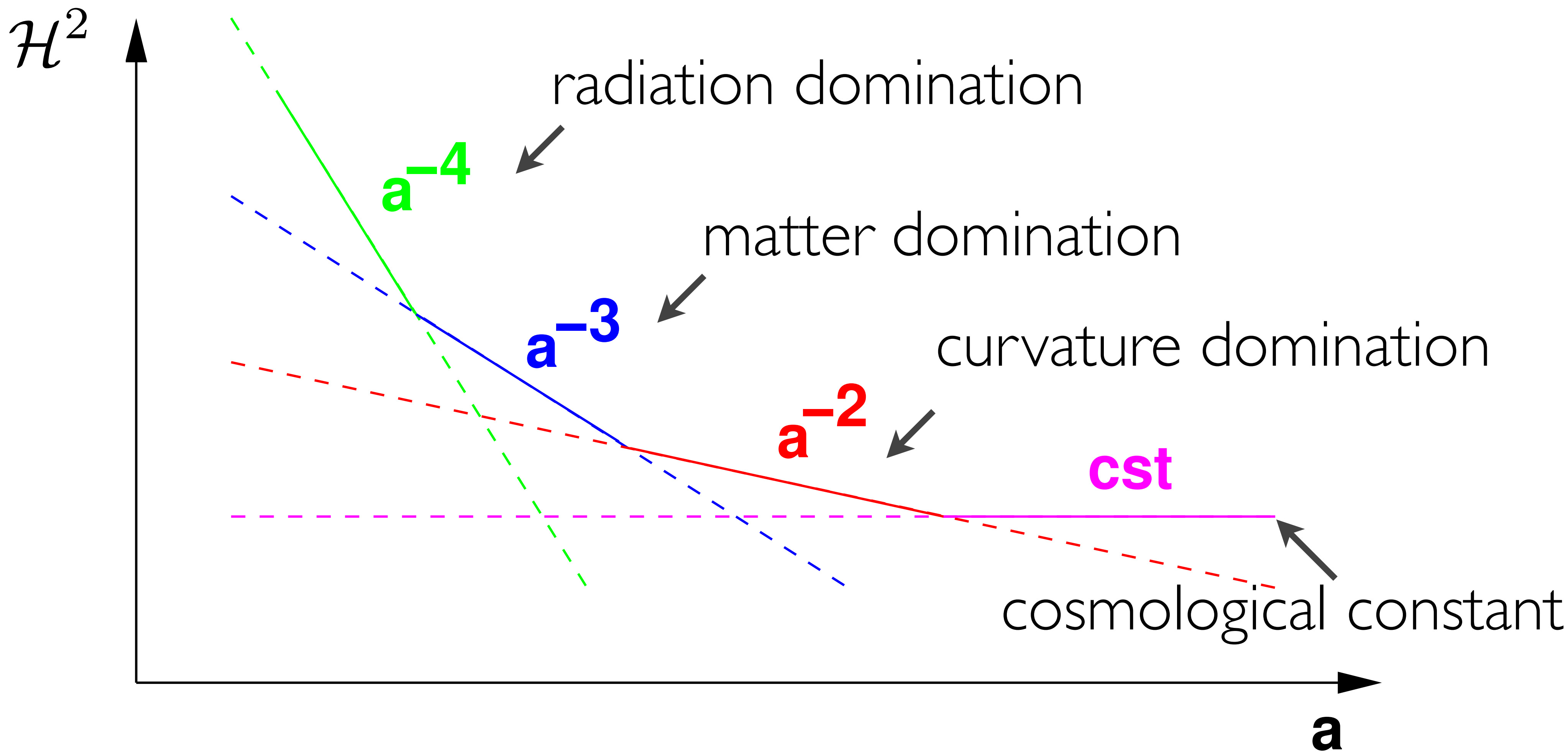
$$\frac{d}{dt}\mathcal{H} = -\frac{4\pi}{3} G a^2 \sum (\bar{\rho}_n + 3\bar{P}_n)$$

matter, dark matter,
dark energy, (m)neutrinos, photons

fast implementation (e.g. in CLASS)
matter sector can be tricky
(e.g. recombination, imperfect fluids,…)

Theoretical framework

i) Background evolution (uncoupled species)



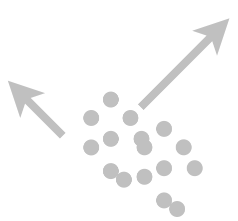
Theoretical framework

ii) Dynamics over the background

Dark Energy $\bar{\rho} = -\bar{P}$

Simplest hypothesis (cosmological constant) is that it has no dynamics

The rest of species: Boltzmann equation



Distribution function in phase space

$$f(x,p,t)d^3xd^3p = dN$$

number of particles in d^3xd^3p

isotropic and homogeneous part

$$f(x,p,t) = f_0(|p|,t)(1+\delta f)$$

$$T_{\mu\nu}(x) = \int d^3p\sqrt{-g}\frac{p_\mu p_\nu}{p^0} f(x,p,t) = \langle T_{\mu\nu}\rangle(t) + \delta T_{\mu\nu}$$

Theoretical framework

Standard definitions

$$\delta T^0_0 = -\delta\rho$$
$$\delta T^0_i = (\bar\rho + \bar P) v_i$$
$$\delta T^i_j = \delta^i_j \delta P + \Sigma^i_j$$

and

$$\delta \equiv \frac{\delta\rho}{\bar\rho} - 1$$
$$(\bar\rho + \bar P)\theta \equiv i k^j \delta T^0_j$$
$$(\bar\rho + \bar P)\sigma \equiv -\left(\hat k_i \hat k_j - \frac{1}{3}\delta_{ij}\right)\Sigma^i_j$$

Boltzmann equation

(phase-space conservation)

$$\frac{Df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial x^i}{\partial t}\frac{\partial f}{\partial x^i} + \frac{\partial p^i}{\partial t}\frac{\partial f}{\partial p^i} = \left(\frac{\partial f}{\partial t}\right)_C$$

(changes number of particles)

The 'exact' solution requires tracing all particles

$$p^0 \frac{dp^\mu}{dt} + \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta = 0$$

(e.g. for massive particles $p^\mu \equiv m \frac{dx^\mu}{ds}$)

non-relativistic: $\frac{\mathrm{d}p^i_A}{\mathrm{d}t} = -a m_A \partial_i \phi$

Theoretical framework

The idea is to deal with 'field' quantities (moments)

Energy density $$\int \mathrm{d}^3p f(x,p,t)\, p^0 \equiv \rho(x,t)$$

Velocity field $$\int \mathrm{d}^3p \frac{p^i}{am} f(x,p,t) \equiv \rho(x,t)\, v^i(x,t)$$

Velocity dispersion $$\int \mathrm{d}^3p \frac{p^i p^j}{(am)^2} f - \rho\, v^i v^j \equiv \sigma^{ij}$$

$\ldots$

They satisfy a chain of hierarchical equations

$$\dot{\rho} = F(v,\rho),\;\; \dot{v}^i = G(v,\rho,\sigma,\ldots)$$

For the different **species** different approximations simplify this analysis

....

Last comments

$$\delta_n \equiv \frac{\rho_n(x,t)}{\bar{\rho}_n(t)} - 1 \qquad \theta = \partial_i v^i \qquad \omega^i = \epsilon^{ijk}\partial_j u^k$$

$\omega^i \sim a^{-1}$ vorticity decays in expanding bckgr.

gravity fields satisfy linear equations, e.g. subhorizon $\Delta\phi = \frac{3}{2}\mathcal{H}^2 \sum \Omega_n \delta_n$

Theoretical framework

Photons $\bar{\rho} = 3\bar{P}$

Treatment **linear** in δf (no cluster)

Coupled to ions and electrons

hierarchy organised in *k* powers

$$\dot{\delta}_\gamma = -\frac{4}{3}\theta_\gamma + 4\dot{\phi}\,,$$

$$\dot{\theta}_\gamma = k^2\left(\frac{1}{4}\delta_\gamma - \sigma_\gamma\right) + k^2\psi + a n_e \sigma_T(\theta_b - \theta_\gamma)\,,$$

→ coupling with ions

$$\dot{F}_{\gamma\,2} = 2\dot{\sigma}_\gamma = \frac{8}{15}\theta_\gamma - \frac{3}{5}kF_{\gamma\,3} - \frac{9}{5}a n_e \sigma_T \sigma_\gamma + \frac{1}{10}a n_e \sigma_T\left(G_{\gamma\,0} + G_{\gamma\,2}\right)\,,$$

$$\dot{F}_{\gamma\,l} = \frac{k}{2l+1}\left[lF_{\gamma\,(l-1)} - (l+1)F_{\gamma\,(l+1)}\right] - a n_e \sigma_T F_{\gamma\,l}\,, \quad l \geq 3$$

$$\dot{G}_{\gamma\,l} = \frac{k}{2l+1}\left[lG_{\gamma\,(l-1)} - (l+1)G_{\gamma\,(l+1)}\right] + a n_e \sigma_T\left[-G_{\gamma\,l} + \frac{1}{2}\left(F_{\gamma\,2} + G_{\gamma\,0} + G_{\gamma\,2}\right)\left(\delta_{l0} + \frac{\delta_{l2}}{5}\right)\right]$$

hierarchy of moments of δf

$$\frac{\int p^2 dp\, \delta f}{\int p^2 dp f} \equiv \sum_{l=0}^{\infty} (-i)^l (2l+1) F_{\nu\,l}(\vec{k},\tau) P_l(\hat{k}\cdot\hat{n})$$

(G is the difference of polarizations)

Equations solved efficiently by Boltzmann codes (they generate the CMB!)

Theoretical framework

Baryons $\bar{P} \approx 0$

As the rest, produced in adiabatic mode (small dispersion)

Coupled to photons before recombination

$$\dot{\delta}_b = -\theta_b + 3\dot{\phi}\,,$$
$$\dot{\theta}_b = -\frac{\dot{a}}{a}\theta_b + c_s^2 k^2 \delta_b + \frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b} a n_e \sigma_T (\theta_\gamma - \theta_b) + k^2\psi$$

hierarchy cut for non-relativistic species starting in adiabatic mode (slow, small dispersion, coupled to ph)

They cluster, but perturbation theory remains valid at large scales

also they are subdominant wrt dark matter clustering

They have internal dynamics at small scales:
e.g. they have a Jeans scale associated to

$$c_s^2 = \frac{\dot{P}_b}{\dot{\rho}_b} = \frac{k_{\rm B} T_b}{\mu}\left(1 - \frac{1}{3}\frac{d\ln T_b}{d\ln a}\right)$$

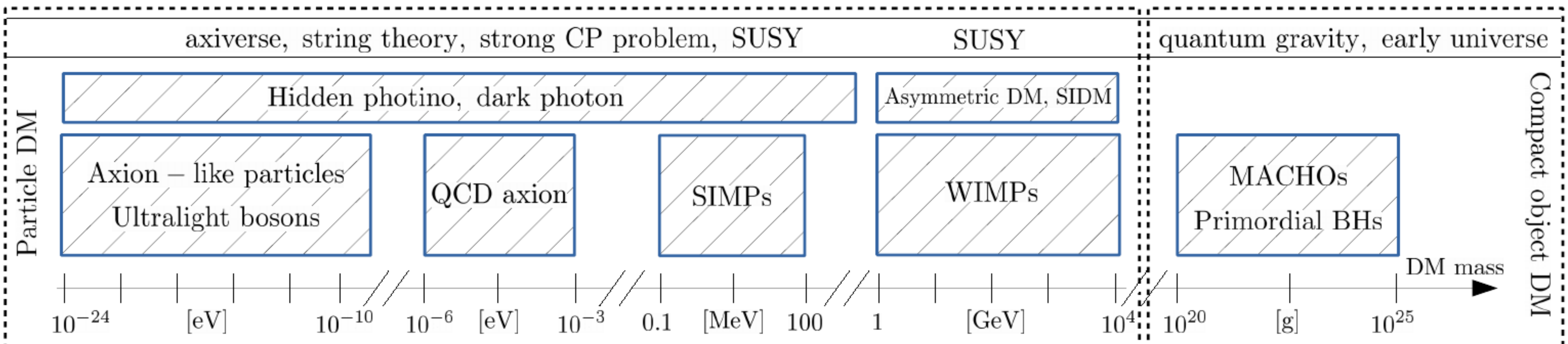
Theoretical framework

Dark matter $\bar{P} \approx 0$

The evidence for DM requires a substance that clusters at different scales and generates halos

$T^{\mu\nu}$ of what? particles? condensate? modified gravity?

which are its 'material' properties? for instance the **mass**…



generating mechanism

Non-thermal generation   keV   Thermal relic   ???

from uncertainty principle   from Ly-a (ask Font-Ribera)/Tremaine-Gunn for fermions

Theoretical framework

Cold DM

'Particles' interacting through gravity and with small velocity dispersion

$$p_A^i \equiv a m_A v_A^i \,, \qquad \frac{\mathrm{d}p_A^i}{\mathrm{d}t} = -a m_A \partial_i \phi$$

Deviation from PPF (dispersion)

$$\dot\delta + \partial_i([1+\delta]v^i) = 0$$

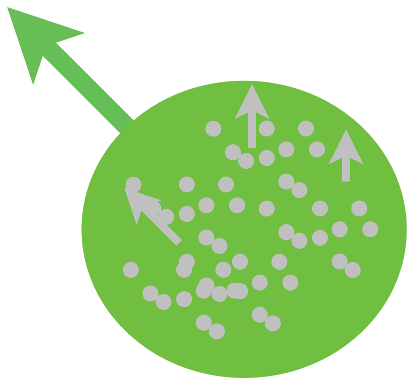
$$\dot v^i + \mathcal{H} v^i + v^j \partial_j v^i = -\partial_i \phi - \frac{1}{\rho}\partial_j \left[\int \mathrm{d}^3 p \frac{p^i p^j}{(am)^2} f - \rho\, v^i v^j \right]$$

Non-linearity

$$v^i(k_1) k_2^i v^j(k_2) \delta^{(3)}(k - k_1 - k_2)$$

This is important when the perturbations grow

Suppressed by $k v_p \mathcal{H}^{-1}$



Linear growth for CDM

Subhorizon

$\mathcal{H}^2 \ll k^2$

$$\dot{\delta} + \partial_i((1 + \cancel{\delta})v^i) = 0$$

$$\dot{v}^i + \mathcal{H}v^i + \cancel{v^j \partial_j v^i} = -\partial_i \phi \quad + \text{~~dispersion~~}$$

$$\Delta\phi = \frac{3}{2}\mathcal{H}^2 \sum \Omega_n \delta_n$$

vorticity decays in expanding bckgr.

$$\theta = \partial_i v^i \qquad \omega^i = \epsilon^{ijk}\partial_j u^k \sim a^{-1}$$

It is better to go to Fourier space

$$\ddot{\delta}_k + \mathcal{H}\dot{\delta}_k = \frac{3}{2}\Omega_m \mathcal{H}^2 \delta_k$$

friction growth!

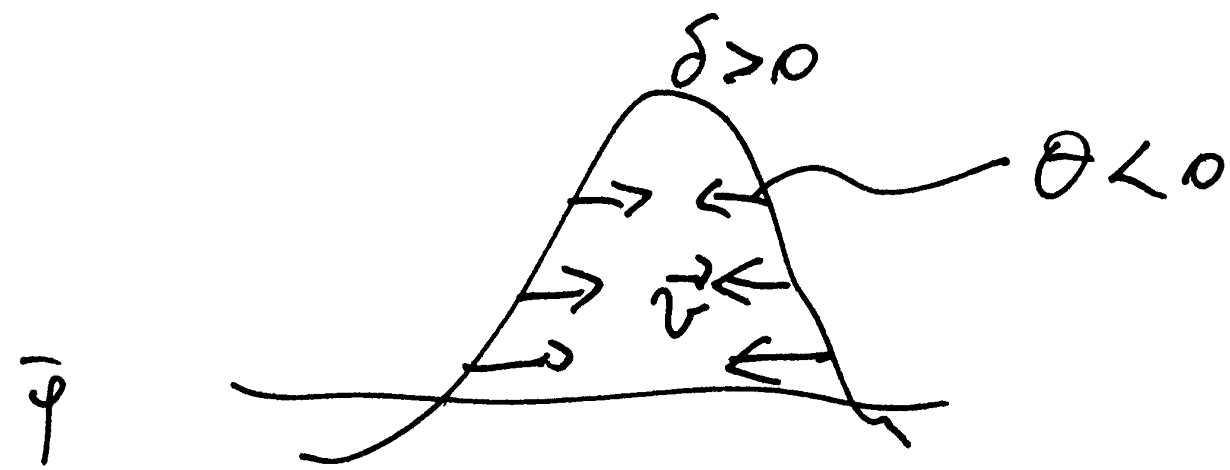
Linear growth for PF

Solution at large scales for DM domination $\Omega_m = 1$

$$\delta_k = A_k a + B_k a^{-3/2}$$

$$\theta_k = -\mathcal{H}\left(A_k a - \frac{3}{2} B_k a^{-3/2}\right)$$

growing mode!



during radiation domination the overdensities don't grow

Stochastic random fields

The comparison with data is done at a statistical level:
e.g. what is the probability that two dense regions are
at a certain distance at a certain time? **(correlation function)**

$$\langle \delta(x)\delta(x+r)\rangle = \xi(r) \qquad \text{(homogeneity)}$$

ergodic assumption: ensemble average = spatial average
and consider $\delta(x,t)$ as random variables

Power spectrum

$$\langle \delta_k(t)\delta_{k'}(t)\rangle = 2\pi^2\delta^{(3)}(k+k')P(k)$$

the distribution was very Gaussian (all the information is
in the PS) at primordial times (inflation)

A lot of information (about dynamics, other initial
conditions,...) in higher order correlations

(homogeneity)

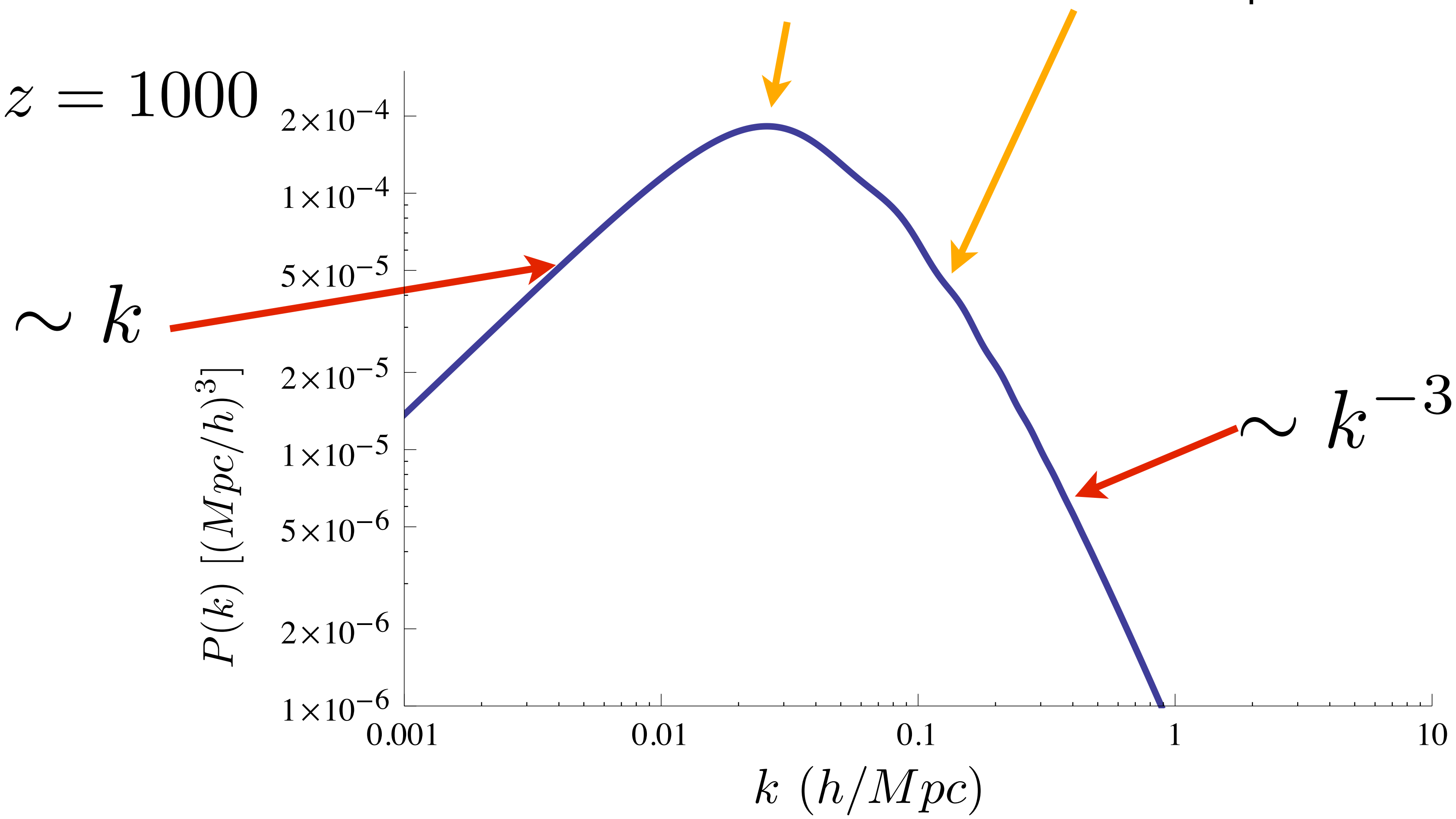
$$\langle \delta_{k_1}\delta_{k_1}\delta_{k_1}\rangle = \delta^{(3)}(\sum k)B(k_1,k_2)$$

Matter power spectrum at decoupling

$$\langle \delta_k(t)\delta_{k'}(t)\rangle = 2\pi^2\delta^{(3)}(k+k')P(k)$$

gaussian initial scale invariant PS + radiation-matter transition + BAO imprint

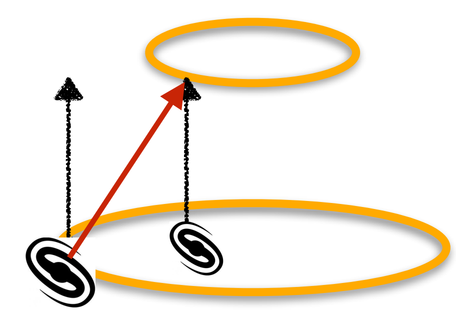
$z = 1000$



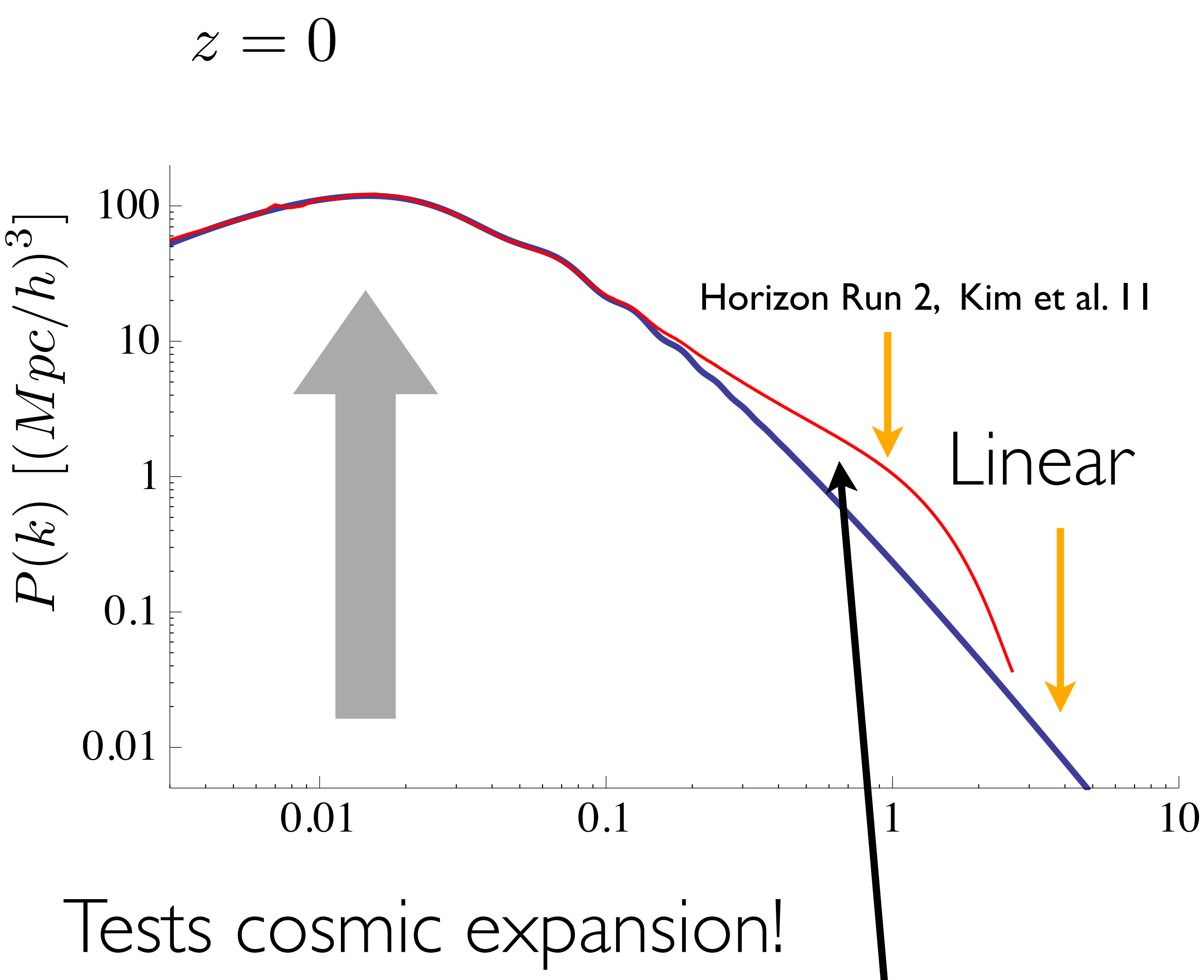
$$P_k \sim \frac{k}{(1+k^2/k_0^2)^2}$$

CLASS CMB Code
Blas, Lesgourgues, Tram 11

unequal times is less about clustering

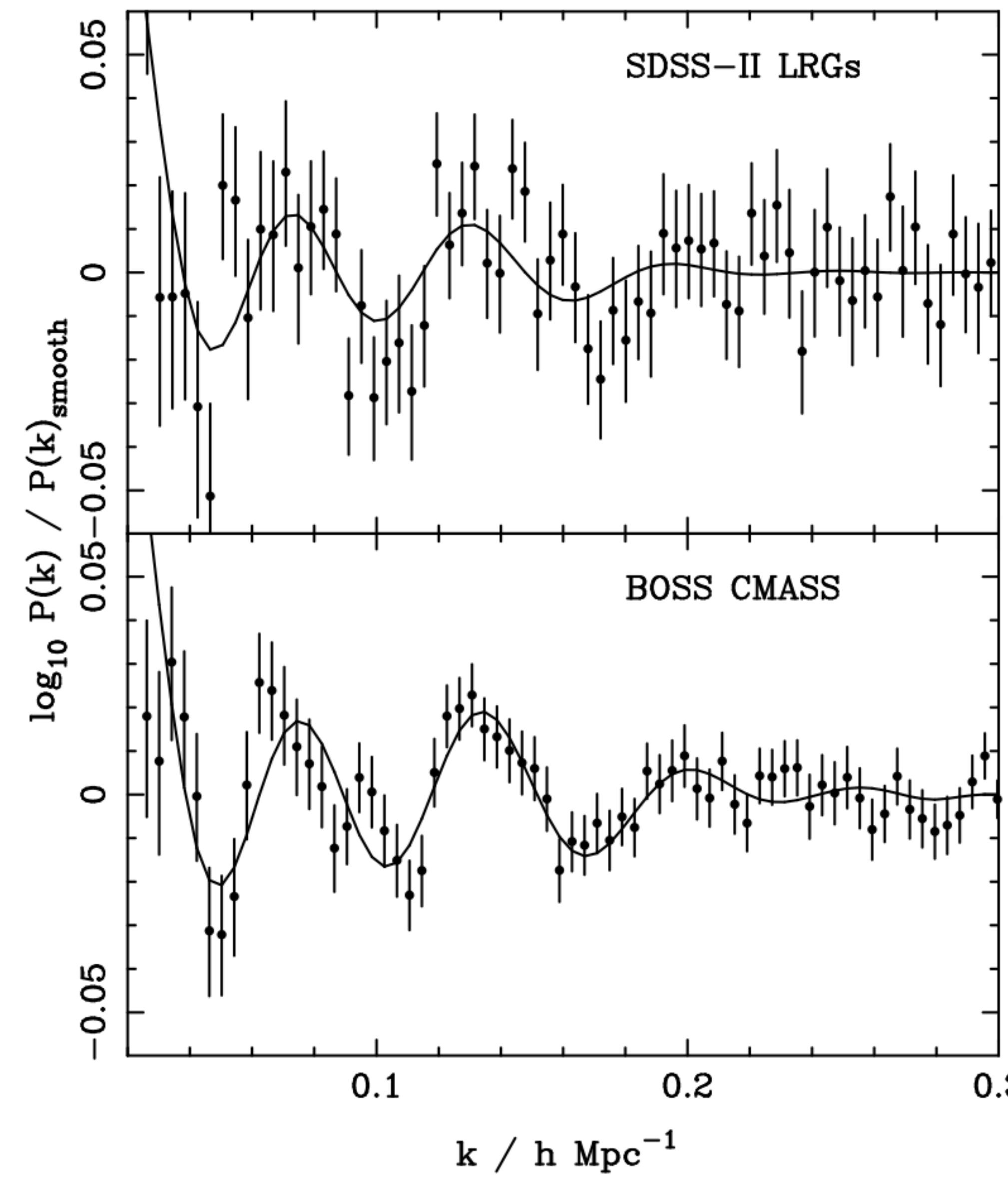


Linear vs Non Linear PS vs Observations



Tests cosmic expansion!
$\delta_k \sim a(t)$

Anderson et al. 12



Failure is inevitable:
perturbations grow + we ignored the dispersion

at which scales? at which precision? (e.g. baryonic effects)

Beyond linear theory: General Statements

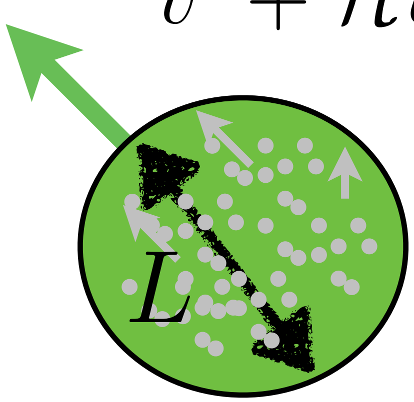
$$v^j \partial_j v^i \quad , \quad \frac{\partial_j}{\rho}\left[\int \mathrm{d}^3 p \frac{p^i p^j}{am)^2} - \rho v^i v^j\right]$$

Coarse-grain at a scale *L*

Effective descr. (large distances) **L** Full treatment? (Kinetic)

One 'solves' the small distances and keeps the large ones

$$\dot{v}^i + \mathcal{H} v^i + v^j \partial_j v^i = -\partial_i \phi - \frac{1}{\rho}\partial_j \left[\int \mathrm{d}^3 p \frac{p^i p^j}{(am)^2} f - \rho\, v^i v^j\right]$$



$$\dot{v}^i_L + \mathcal{H} v^i_L + v^j_L \partial_j v^i_L = -\partial_i \phi_L + \mathcal{O}_L \mathcal{O}_S + \mathcal{O}_S + \mathcal{O}_{mf}$$

Effective approach: encapsulate these effects in operators of ***L***

$$\dot{v}^i_L + \mathcal{H} v^i_L + \partial_L \phi = -v^i_L \partial_j v^i_L + c(t)^2_{L;s} \partial_i \delta_L + \frac{c(t)^2_{L;v}}{\Lambda(H, k_s)} \partial_i \partial_j v^j_L + \ldots + \mathcal{O}_{stoch}$$

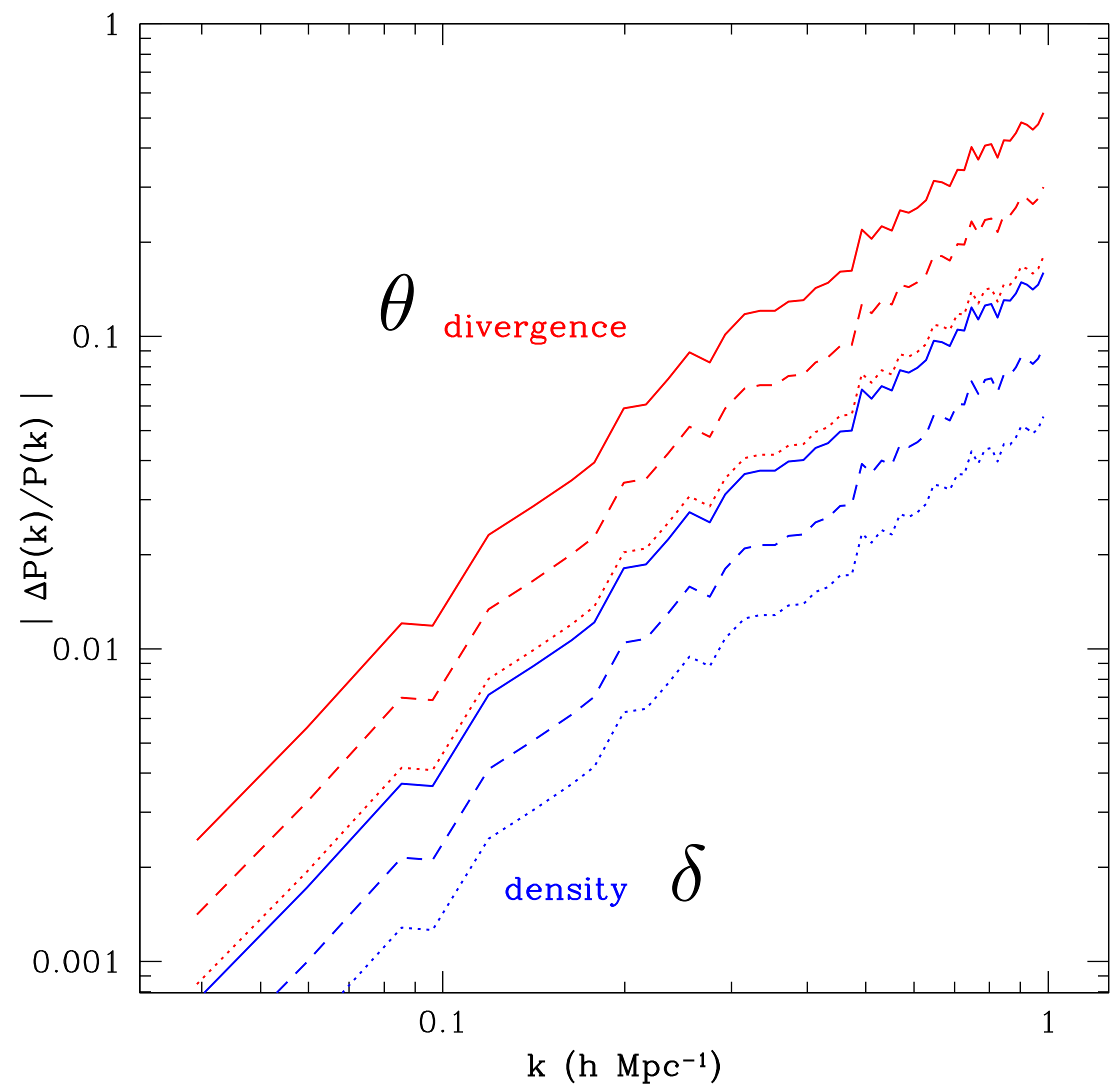
Pietroni et al 11, Carrasco et al 12

perfect fluid 'imperfect' fluid stochastic term

Effects of coefficients from small scales

Pueblas, Scoccimarro 09

$$\dot{\theta}(k,t) + \mathcal{H}\theta(k,t) + \frac{3}{2}\Omega_m \mathcal{H}^2 \delta(k,t) = q_\theta(k,t)$$



Negligible at .2!

• •

More on the coefficients and NL equations in Martin's talk

Theoretical framework

Neutrinos

They are massive but with tiny masses

Laboratory constraints

$$0.06 \text{ eV} < \sum_{\psi=e,\mu,\tau} m_{\nu_\psi} \quad , \quad \sum_{\psi=e,\mu,\tau} \alpha_{\nu_\psi} m_{\nu_\psi} < 2 \text{ eV} \ (95\% \ CL)$$

Cosmology constraints (more on Pastor/Mena talks)

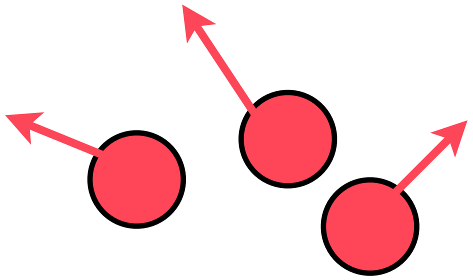
$$\sum m_i < 0.14 \ (95\%)$$

 ?

they are produced relativistic, behave as photons (except for the coupling to matter), get cold and then behave as dark matter (cluster, non-linear)

Their treatment requires more care than for other species

Description of neutrinos



thermal background $f_{\nu 0}(\eta, p) \equiv \left(e^{p/T_\nu} + 1\right)^{-1}$

(linear) Boltzmann equation $(E(p) = \sqrt{p^2 + m_\nu^2})$

Massless neutrinos $E(p) = p$ free-stream and do not cluster

$$\delta\rho_\nu = \int d^3p E(p) f_\nu(\eta, \bar{x}, \bar{p}) \,, \qquad \delta\rho_\nu(k)'' = (c^2(\eta)k^2 - 3a^2H^2/2)\delta\rho_\nu(k) + \ldots$$

($c^2(\eta)$ ↖ $1/3$)

efficiently treated in Boltzmann codes

Massive: when $p \ll E(p) \sim m$, neutrinos become **cold (cluster)**

$$k_{fs} \sim aH/c \quad , \qquad c(\eta) \sim T_\nu(\eta)/m_\nu$$

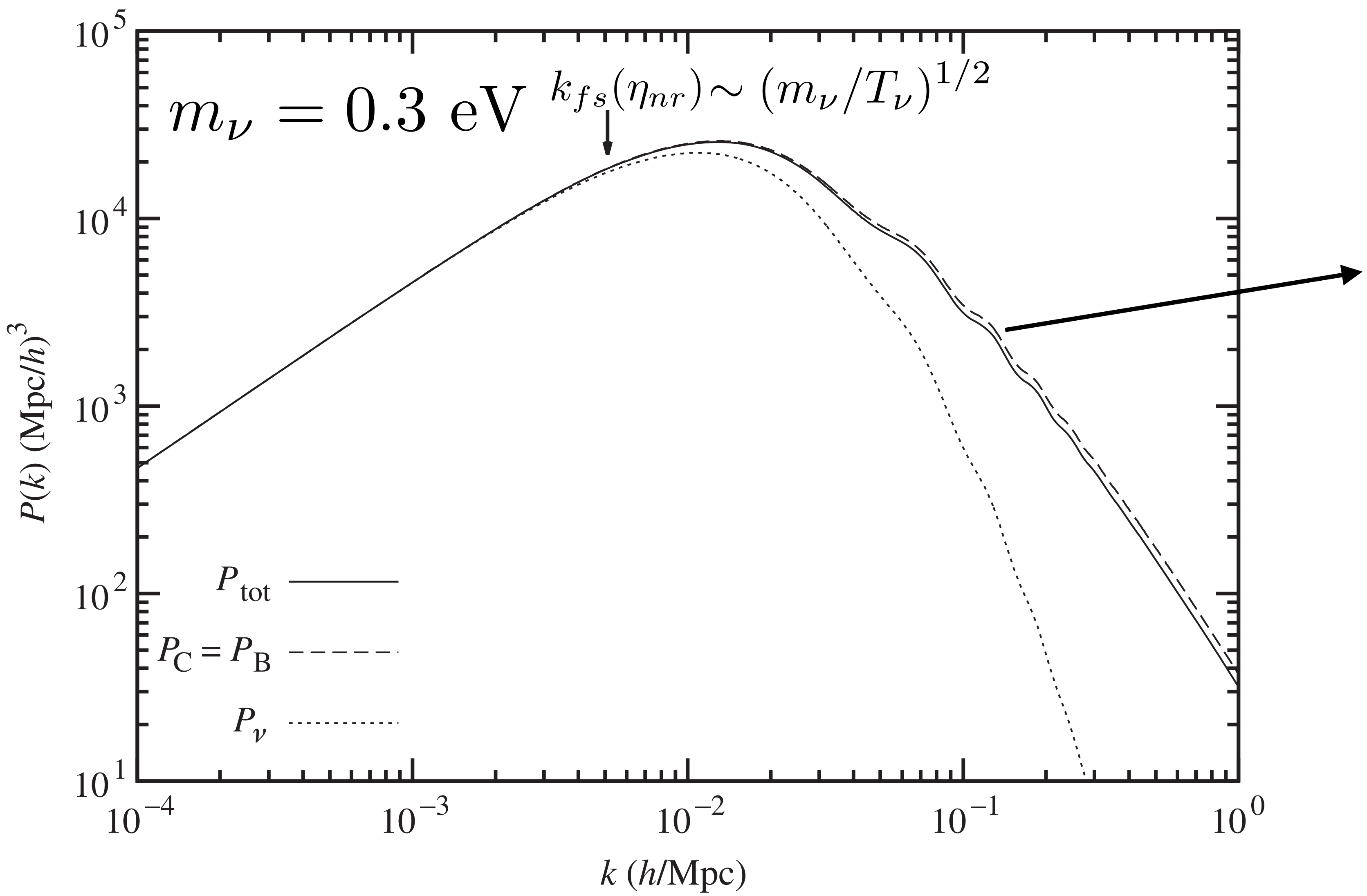
they behave as DM!

known law e.g. Shoji, Komatsu 2010

Effects of m_ν on the linear power spectrum

$$\delta \equiv \frac{\sum_{i=b,c,\nu} \delta\rho_i}{\sum_{i=b,c,\nu} \bar{\rho}_i}$$

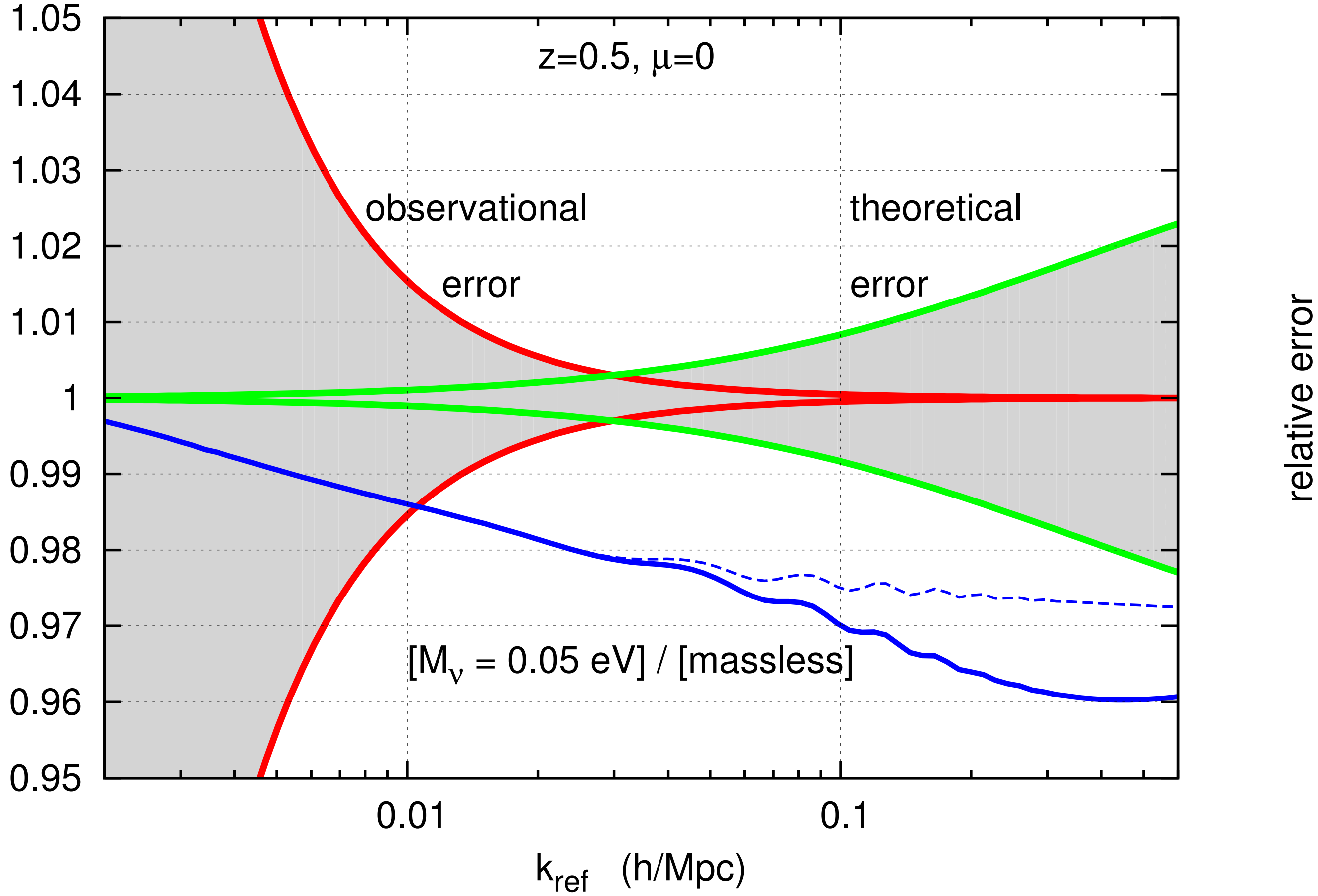
Lesgourgues, Mangano, Miele, Pastor CUP 2013



at least
5% effect

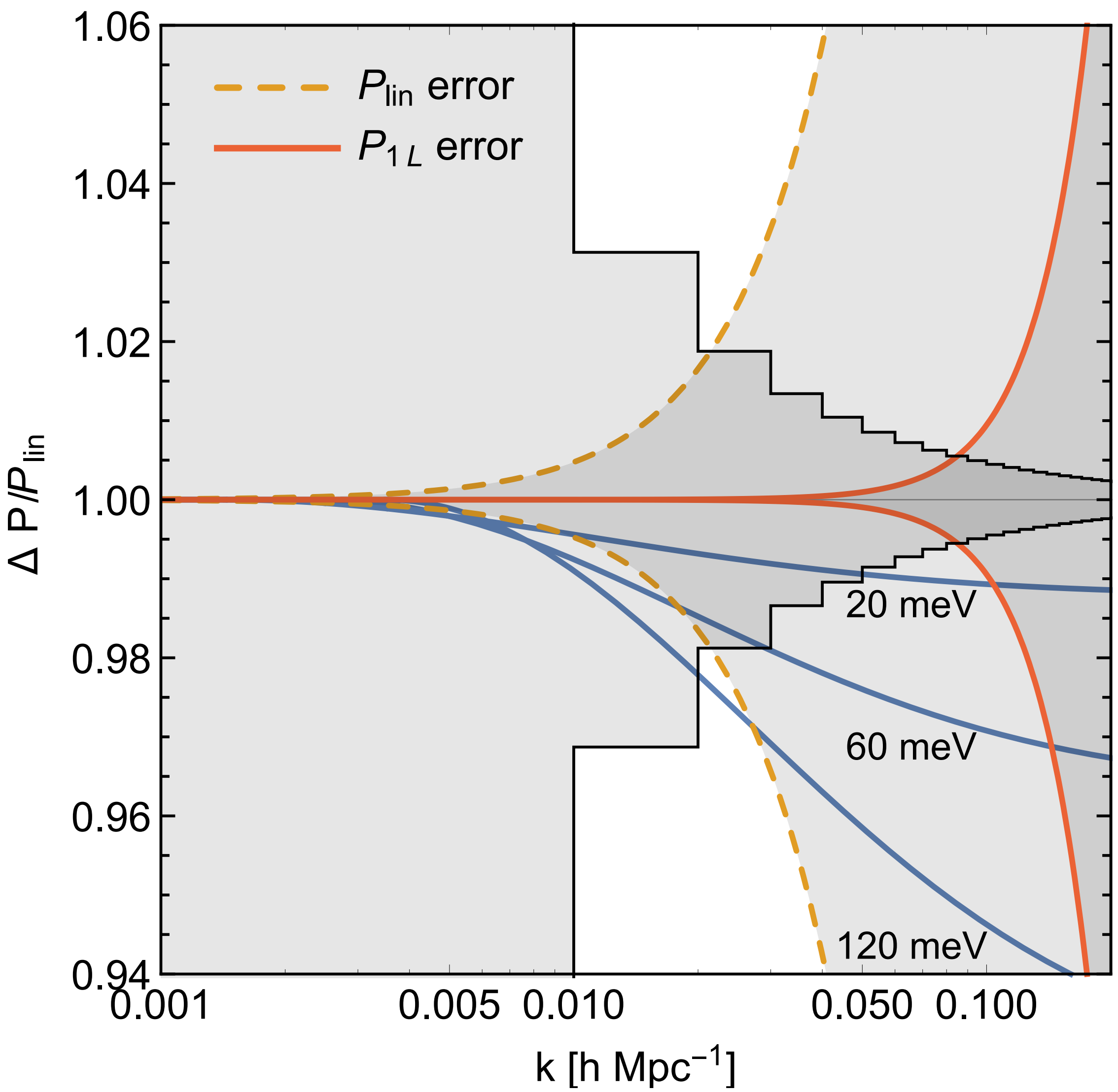
Audren et al. 2013

Is it enough?



Audren et al. 2013 EUCLID Forecast
based on HALOFIT

Baldauf et al 2016



Beyond linear theory

N-body (with warm components)
Demanding (hard for MC)
Halo model (~10% precision)

The effect is 5% at BAO scales
(mildly non-linear regime):
Non-linear perturbation theory

DB, Garny, Konstandin, Lesgourgues'14
(also Führer-Wong'14, Dupuy-Bernardeau'14
Archidiacono-Hannestad'15)

DM as a **non-linear** pressureless perfect fluid
(SPT or 'beyond')

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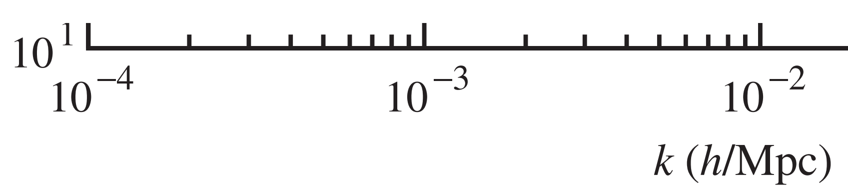
Neutrinos

CDM-like

Phase-space



$$\Delta\phi = \frac{3}{2}\mathcal{H}^2\Omega_m[f_\nu\delta_\nu + (1-f_\nu)\delta_{cb}]$$

$$f_\nu \equiv \frac{\Omega_\nu}{\Omega_m} = \frac{1}{\Omega_m^0 h^2}\frac{\sum_i m_{\nu,i}}{93.14\,\mathrm{eV}}$$

Linear vs Non-linear ν's II

How to include ν non-linearities?

(even linear order is **NOT** a fluid at **all redshift**)

Shoji, Komatsu 2009

Blas, Lesgourgues, Tram 2011

At low-redshift ($z < z_{nr} \sim 10^2$) the fluid is very cold

non-cold corrections are $O(T_\nu/m_\nu)$

Neutrinos at late times

$$\dot{\delta}_\nu + \theta_\nu = -\alpha\theta_\nu\delta_\nu$$

$$\dot{\theta}_\nu + \mathcal{H}\theta_\nu + \frac{3}{2}\mathcal{H}^2\Omega_m[f_\nu\delta_\nu + (1-f_\nu)\delta_{cb}] - k^2 c_s(t)^2\delta_\nu = -\beta\delta_\nu\theta_\nu + O(T_\nu/m_\nu)$$

i.c. from the Boltzmann equations at $10 > z > 10^2$

linear physics

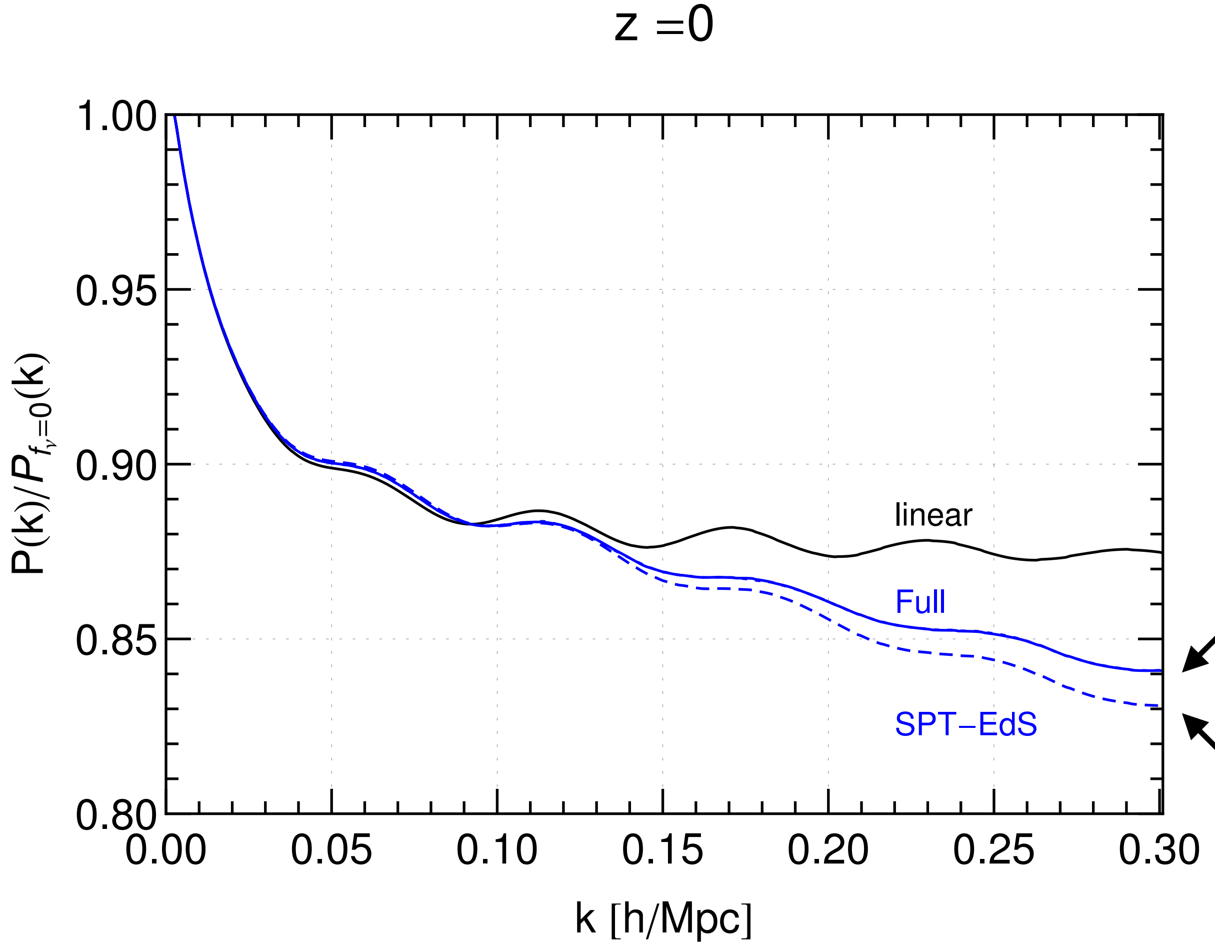
Results at NLO

DB, Garny, Konstandin, Lesgourgues 2014

Pietroni 2008

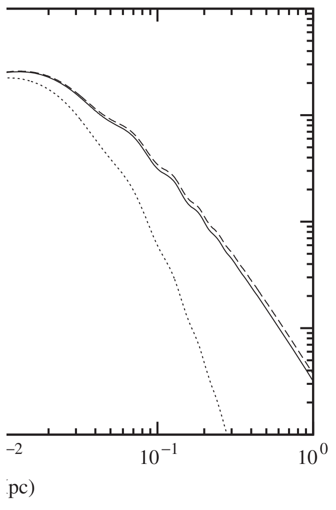
Audren Lesgourgues 2011

Scale dependent growth factor



two fluids

one fluid with suppressed PS



Conclusions

- Evolution of the background 'easy' for the different components
- Perturbations over the background are produced by different components with different properties: Boltzmann equation
  - moments of the Boltzmann equation allow for analytical treatment
  - the typical approximations are: linearity and small dispersion (coldness of the medium)
  - photons, baryons, dark energy: treated efficiently
  - dark matter and massive neutrinos are more complicated: they cluster (~~linearity~~), and neutrinos are hot at early stages
- These methods allow to get the fingerprints of the universe. They can be extended to study new properties (Zumalacarregui's talk)

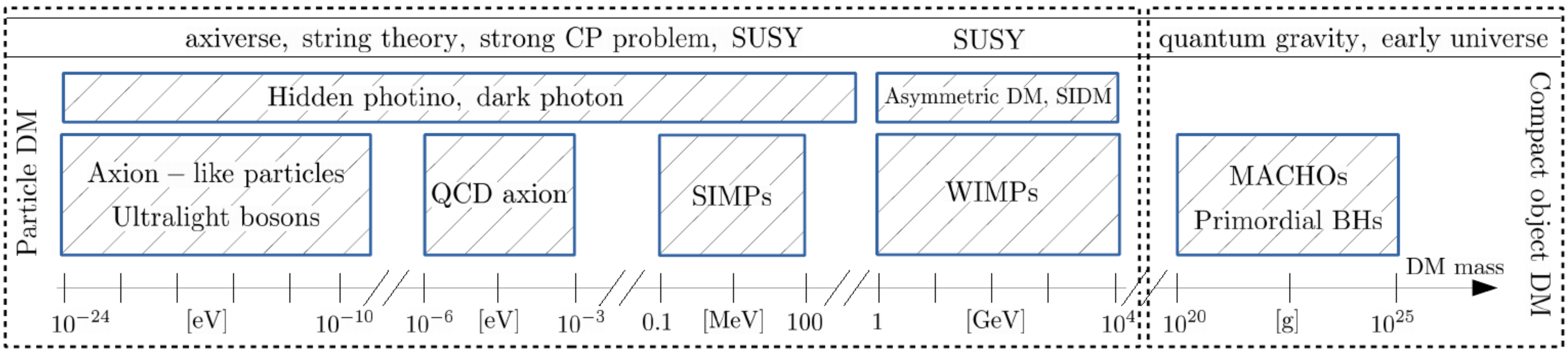
Some goals

Dark energy properties

$$\bar{\rho} = -\bar{P}?$$

new dynamics?

Dark matter properties



Neutrinos mass

$$m_\nu$$

Some tools…

https://arxiv.org/abs/1603.04826

Fast 1 loop

http://sns.ias.edu/matiasz_filedrop/

Many tools for 1 loop calculations