1.) $$\Delta f \simeq \Delta\tau\, C[f_0] = \Delta\tau \frac{\theta_e}{x^2}\partial_x x^4\left[\partial_x f_0 + \frac{T_\gamma}{T_e} f_0(1+f_0)\right]$$

i) need $\partial_x f_0 = ?$

$$\partial_x \frac{1}{e^x-1} = -\frac{e^x}{(e^x-1)^2} \overset{D}{\doteq} -\frac{(e^x-1+1)}{(e^x-1)^2}$$

$$\overset{D}{\doteq} -f_0(1+f_0)$$

$$\Rightarrow \quad \Delta f = \frac{\Delta\tau\,\theta_e}{x^2}\partial_x\left\{x^4\left[\frac{T_\gamma}{T_e}-1\right]f_0(1+f_0)\right\}$$

$$\theta_e = \frac{kT_e}{mc^2}$$

$$= \overbrace{\frac{\Delta\tau(\theta_\gamma-\theta_e)}{x^2}}^{=:-y}\underbrace{\partial_x x^4 f_0(1+f_0)}_{}$$

$$= 4x^3 f_0(1+f_0) + x^4\partial_x f_0(1+f_0)$$

$$\partial_x(f_0+f_0^2) = -f_0(1+f_0) + 2f_0\left[-f_0(1+f_0)\right]$$

$$= -f_0(1+f_0)\left[1+2f_0\right] = -f_0(1+f_0)\frac{e^x+1}{e^x-1}$$

$$\Rightarrow \boxed{\begin{aligned} \Delta f &\simeq y \frac{x e^x}{(e^x-1)^2}\left[x\frac{e^x+1}{e^x-1} - 4\right] \\ y &= \int (\theta_e - \theta_\gamma)\, d\tau \end{aligned}}$$

2a) $C[f] = 0 \iff \partial_x f + \frac{T_\gamma}{T_e} f(1+f) = 0$

$$\partial_x f_{BE} = -f_{BE}(1+f_{BE})\left[1 + \partial_x \mu\right]$$

$$\Rightarrow 1 + \partial_x \mu = \frac{T_\gamma}{T_e} \quad \Rightarrow \mu(x) = \mu_0 + x\left(\frac{T_\gamma - T_e}{T_e}\right)$$

$$\Rightarrow f_{BE} = \frac{1}{e^{x_e + \mu_0} - 1}$$ is general solution

$\mu_0 = 0 \iff$ Blackbody at $T_\gamma = T_e$

2b) $\partial_x f + \frac{T_\gamma}{T_e} f \simeq 0$

$$\Rightarrow \frac{\partial f}{f} = -\frac{T_\gamma}{T_e} d_x$$

$$\Rightarrow \ln f/f_0 = -x_e \Rightarrow \boxed{f(x_e) = f_0 e^{-x_e}}$$

Wien spectrum!

3a) $\frac{dE}{dt} \sim -M_X c^2 \cdot \dot{N}_X \sim M_X c^2 \cdot \Gamma_X N_X$

decay of particles leads to release of energy

$$N_X \sim N_{X,0} \cdot e^{-\Gamma_X t}$$

$$\frac{d}{dt}\left(\frac{Q}{\varepsilon_\gamma}\right) = \frac{M_X c^2 \Gamma_X N_{X,0} e^{-\Gamma_X t}}{\varepsilon_\gamma}$$

$$\frac{d}{dt} \to \frac{d}{dz} \to \frac{1}{H(1+z)}$$

$$\Rightarrow \frac{d(a/g_\gamma)}{dz} = \frac{M_x c^2 \Gamma_x N_{x,0}}{g_\gamma H (1+z)} e^{-\Gamma_x t}$$

$$\overset{!}{=} \varepsilon_x \frac{N_H(z)(1+z_x)}{g_\gamma H (1+z)} \Gamma_x e^{-\Gamma_x t}$$

$$\Rightarrow \varepsilon_x = \frac{M'_x c^2 N_{x,0}}{N_{H,0}(1+z)}$$

3b) $H \sim (1+z)^2$ $\qquad g_0 \sim (1+z)^4$

$N_H \sim (1+z)^3$

$$\Rightarrow \frac{d(a/g)}{dz} \sim \frac{(1+z)^3}{(1+z)^{4+2+1}} e^{-\Gamma_x t} \sim \begin{cases} 1/z^4 & t \ll t_x \\ e^{-\Gamma_x t} & t > t_x \end{cases}$$

---

4)

$$E_{gas} \sim \frac{3}{2} N_{tot} kT$$

$$\frac{dE_{gas}}{dt} \sim \frac{3}{2} \frac{N_{tot} kT}{t_{exp}} \sim \frac{3}{2} N_{tot} kT_\gamma H$$

$$\frac{d(Q/s_\gamma)}{dz} = -\frac{3}{2} \frac{N_{tot} kT_\gamma H}{s_\gamma H (1+z)} = -\frac{3}{2} \frac{N_{tot} kT_\gamma}{s_\gamma (1+z)}$$

more particles $\Rightarrow$ $\boxed{\text{higher cooling}}$

$N_{tot} \uparrow$