Wave diagrams for ideal 2-fluid plasmas

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MHD wave signals

- **static homogeneous plasma**: slow, Alfvén, fast wave pairs

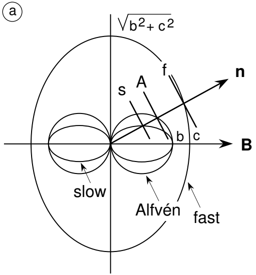
  $\Rightarrow$ the **phase speed diagrams** quantify for every angle ϑ between **k** and **B** how far a plane wave can travel in fixed time

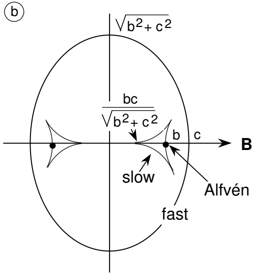
  $\Rightarrow$ point perturbation leads to the related **group diagram**, found from a Huygens construction on the phase speed diagram (constructive interference of all plane waves)

Phase and group diagrams [G&P, CUP, 2004]

Friedrichs diagrams (schematic)

parameter $c_s/b = \frac{1}{2}\gamma\beta\,,\ \beta \equiv 2p/B^2$]

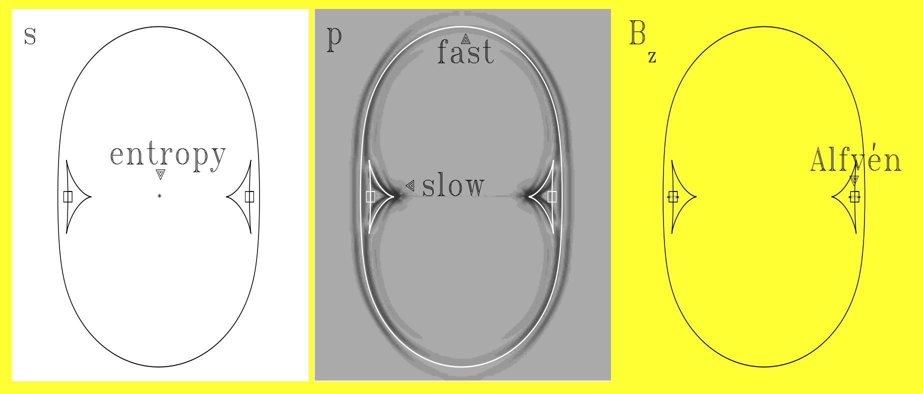




Phase diagram
(plane waves)

Group diagram
(point disturbances)

- locally perturb homogeneous magnetized plasma at rest
  - $\Rightarrow$ $\gamma = 5/3$, $\rho = 1$, $p_{\rm th} = 0.6$ and $\mathbf{B} = 0.9\hat{e}_x$ ($c_s = 1, b = 0.9$)
  - $\Rightarrow$ $(x, y) \in [-0.5, 0.5]^2$ in 2.5D ideal MHD, include v_z, B_z
  - $\Rightarrow$ perturb at origin with $\delta\rho = 0.1$, $\delta v_z = 0.01$ and $\delta p_{\rm th} = 0.06$
- MHD counterpart of **'throwing a stone in a puddle'**



$\Rightarrow$ entropy, total pressure, B_z at finite time

Extension to Hall-MHD

- Hall-MHD: ion dynamics (massless e) in charge-neutral plasma $n_\mathrm{e} = Zn_\mathrm{i}$, where speeds $\mathbf{v} = \mathbf{u}_\mathrm{i}$ and $\mathbf{u}_\mathrm{e} = \mathbf{v} - (en_\mathrm{e})^{-1}\mathbf{j}$ and $\rho = n_\mathrm{i} m_\mathrm{i}$

  $\Rightarrow$ induction equation modifies to:

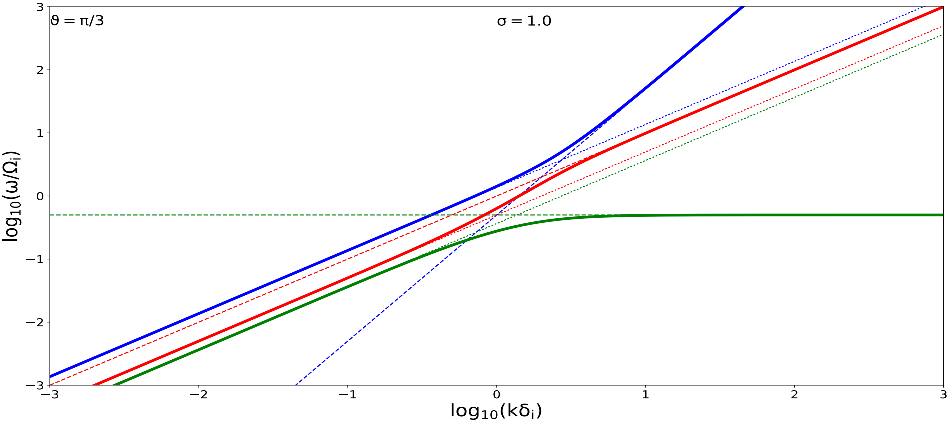
$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times [(\mathbf{v} - \frac{m_i}{Ze\rho}\mathbf{j}) \times \mathbf{B}] = \mathbf{0}$$

  $\Rightarrow$ introduces ion inertial length $\delta_\mathrm{i} \equiv c/\omega_\mathrm{pi}$, obtain DR

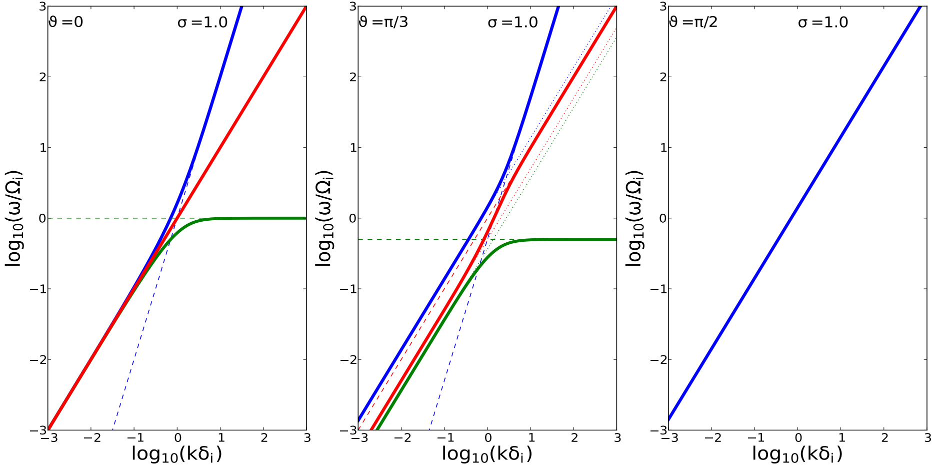
$$(\omega^2 - k_\parallel^2 b^2)\left[\omega^4 - k^2(b^2 + c_s^2)\omega^2 + k_\parallel^2 k^2 b^2 c_s^2\right] - \lambda_\mathbf{H}\omega^2 \mathbf{k}_\parallel^2 \mathbf{b}^2 \left(\omega^2 - \mathbf{k}^2 \mathbf{c}_\mathbf{s}^2\right) = 0$$

  $\Rightarrow$ waves now dispersive, $\lambda_\mathrm{H} \equiv (k\delta_\mathrm{i})^2$ Hall parameter

- Hameiri et al, PoP 12, 072109 (2005): study DR, vary $\sigma = c_s^2/b^2$
  - $\Rightarrow$ wave normal (phase) and ray surfaces (group)
  - $\Rightarrow$ still 3 pairs of waves (forward-backward)
  - $\Rightarrow$ all 3 waves dispersive: seen in $\omega - k$ diagrams

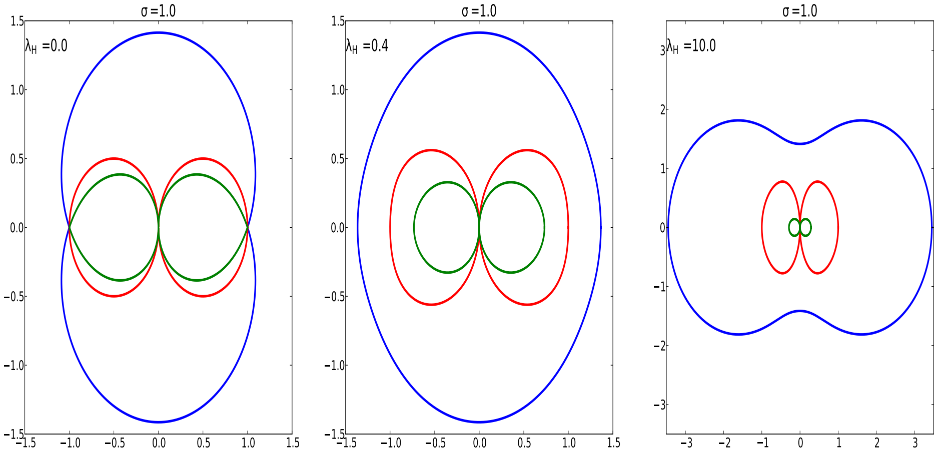


- $\omega - k$ diagrams can be shown for varying ϑ (angle **k** and **B**)



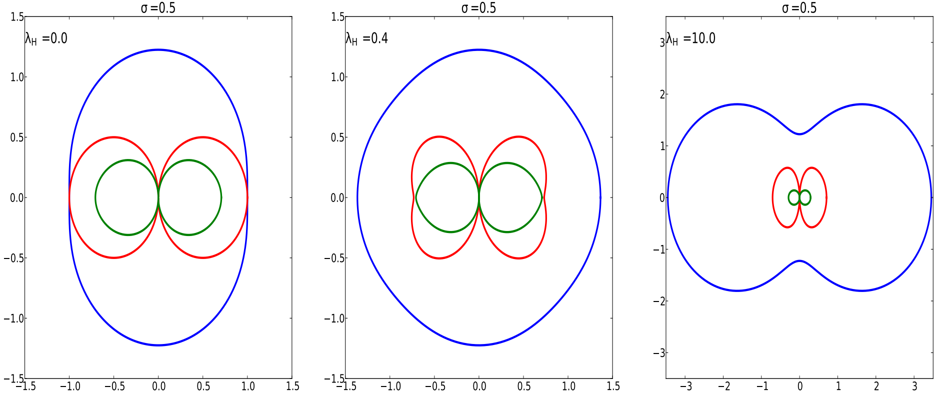
⇒ **from parallel to perpendicular**

- alternative representation: phase diagrams (wave normal surfaces): fix $\lambda_{\rm H} = k\delta_i$, show all angles (left panel is MHD)



- can show this for varying σ (i.e. β) and

animate for varying wavelength (increasing Hall parameter)

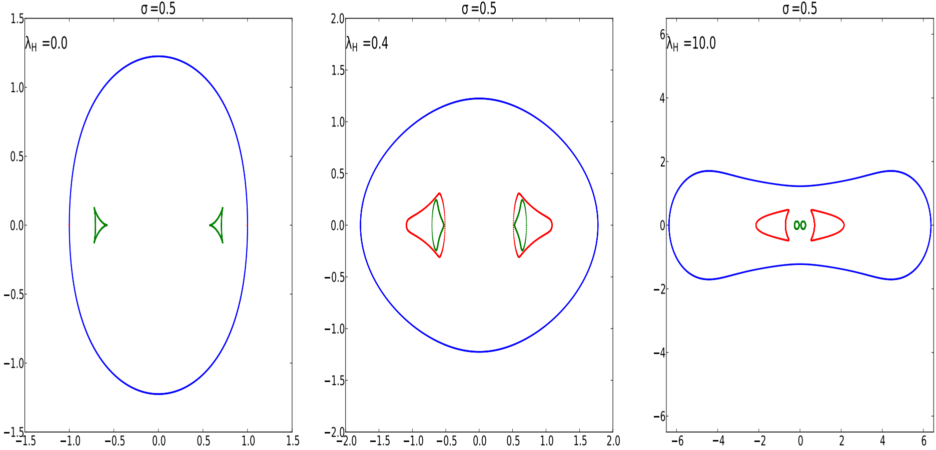


- Much more intriguing: ray surfaces (group diagrams): implicit derivation on DR yields $\frac{\partial \omega}{\partial \mathbf{k}}$ expressions as

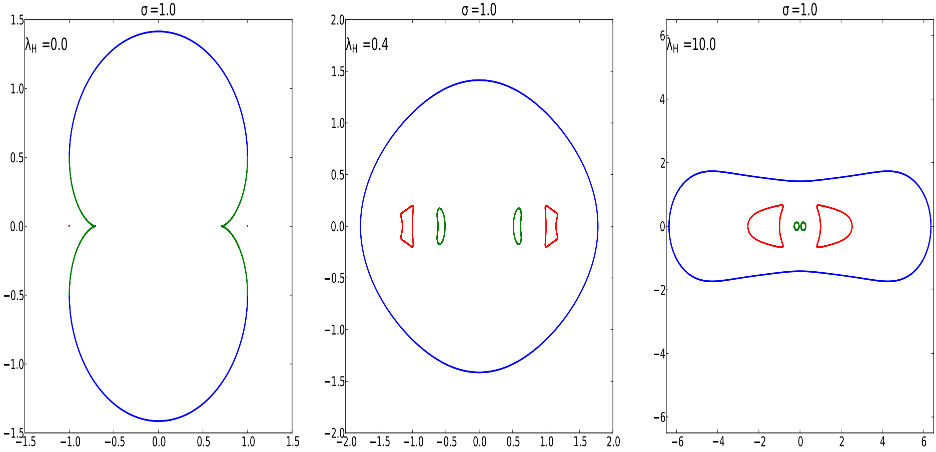
$$\frac{\partial \omega}{\partial \mathbf{k}} = f_b(\omega, \sigma, k, \cos\vartheta)\hat{\mathbf{b}} + f_n(\omega, \sigma, k, \cos\vartheta)\hat{\mathbf{n}}$$

$\Rightarrow$ quantifies approximate wave fronts, $\hat{\mathbf{b}} = \mathbf{B}/B$ and $\hat{\mathbf{n}} = \mathbf{k}/k$

- for $\sigma = 0.5$, MHD to large Hall parameter
  $\Rightarrow$ note the sometimes 'strange' ordering (slow-Alfvén-fast)

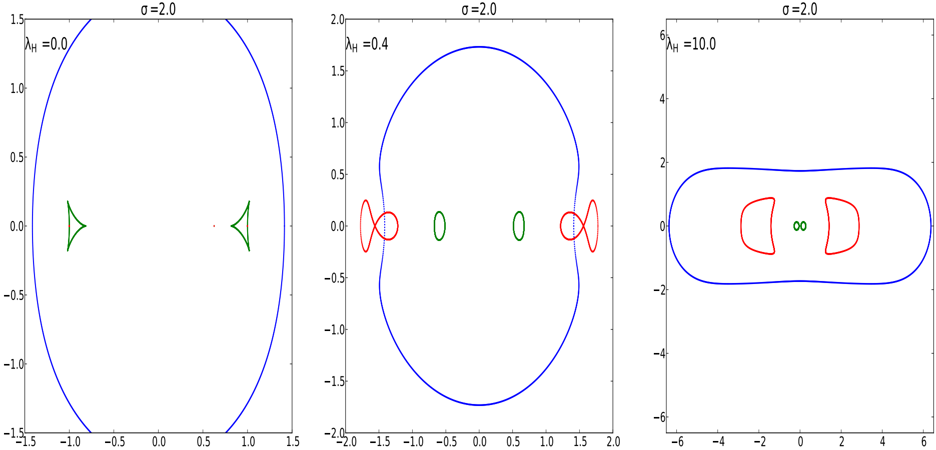


- for $\sigma = 1$, MHD (also special in MHD!) to large Hall parameter

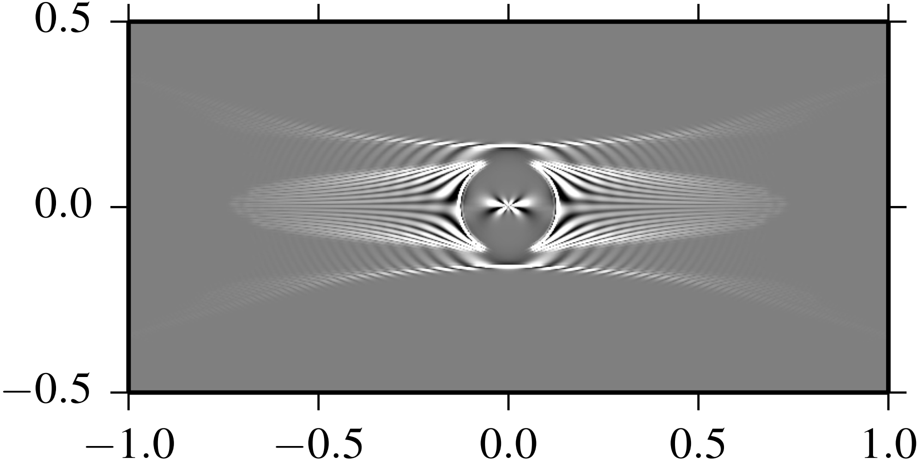


- for $\sigma = 2$, MHD to large Hall parameter

animation (increasing Hall parameter)



- relevant as test for numerical Hall-MHD: Porth et al, ApJS 214, 2014 (MPI-AMRVAC): pressure pattern emerging from interference, fast & Alfvén envelope



Ideal 2-fluid diagrams

- See G&P, 2004 [or new edition coming soon: **GK&P, 2018!**]

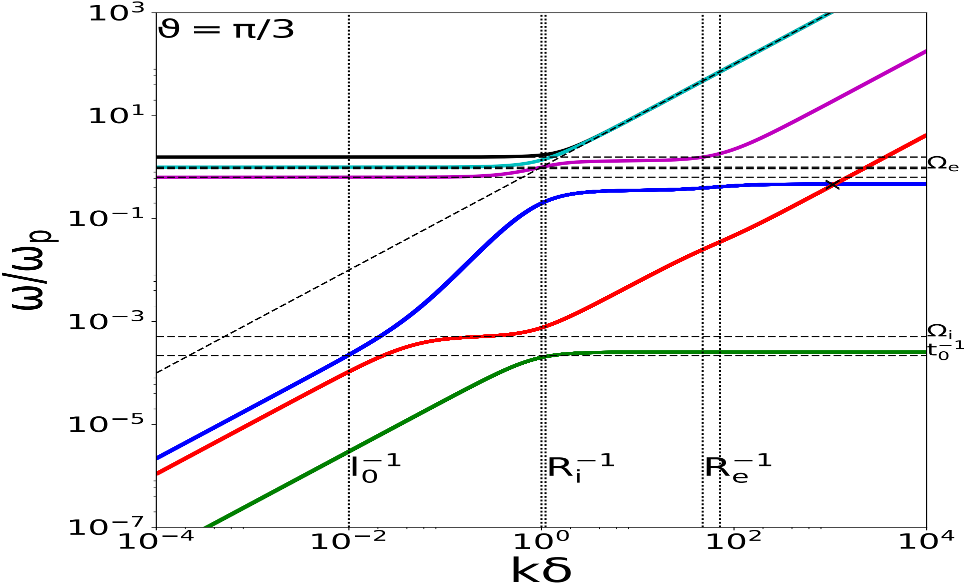
  $\Rightarrow$ DR best written in terms of $\bar{\omega} = \omega/\omega_p$, $\bar{k} = kc/\omega_p = k\delta$ with plasma frequency ω_p and skin depth δ, then obtain 12-th order polynomial (6th order in ω^2, fourth in k^2)

  $\Rightarrow$ parameters $E = \Omega_e/\omega_p$ (electron cyclotron), $v = v_e/c$, $w = v_i/c$ (sound speeds) and $\mu = Zm_e/m_i$ (mass ratio)
- known limits:

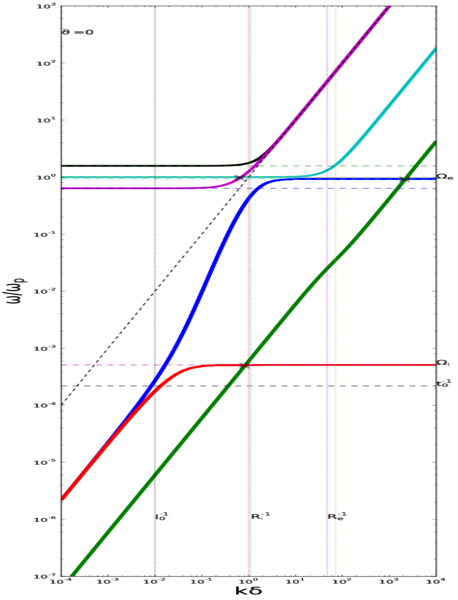
  $\Rightarrow$ short k: MHD and plasma cut-offs

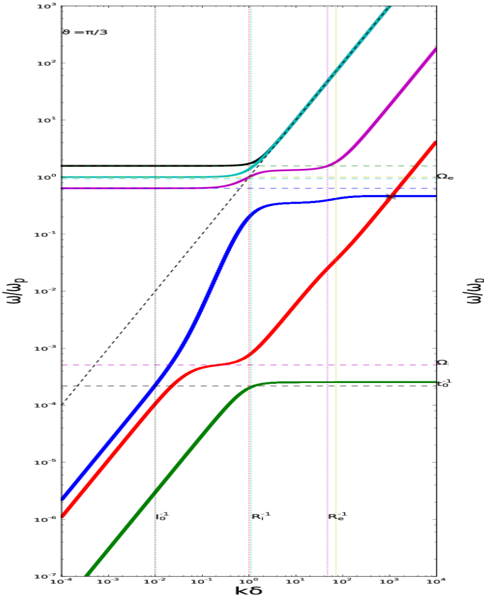
  $\Rightarrow$ large k: 2xEM (kc), ion and e sound, e and ion cycl. res.

- $\omega - k$ diagrams, for varying angles $\cos(\vartheta) = \mathbf{k}\cdot\mathbf{B}/kB$, coronal loop



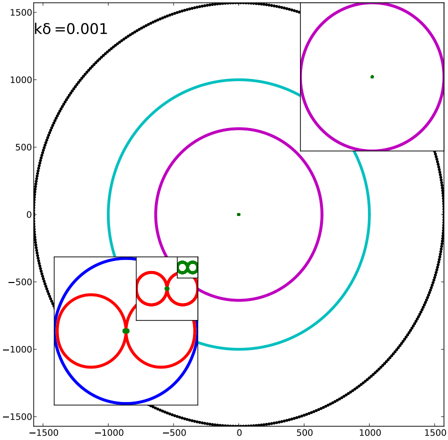
- vary angles $\cos(\vartheta) = \mathbf{k} \cdot \mathbf{B}/kB$, coronal loop





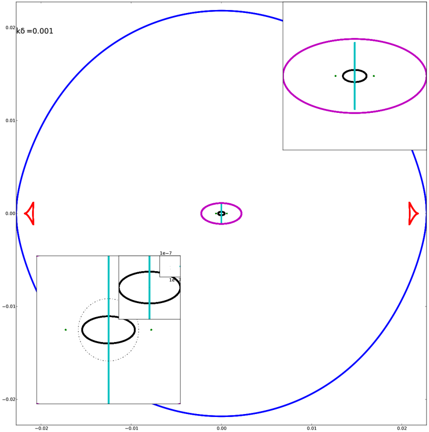
- as angle varies: branches show (avoided) crossings, ‘labeling’ waves must ultimately involve the way eigenfunctions remain similar on various branches!
  see changeover through zeros of derivative of DR

- Show alternative wave normal (phase) diagrams, for varying $k\delta$
  $\Rightarrow$ note several branches with superluminal phase speeds!

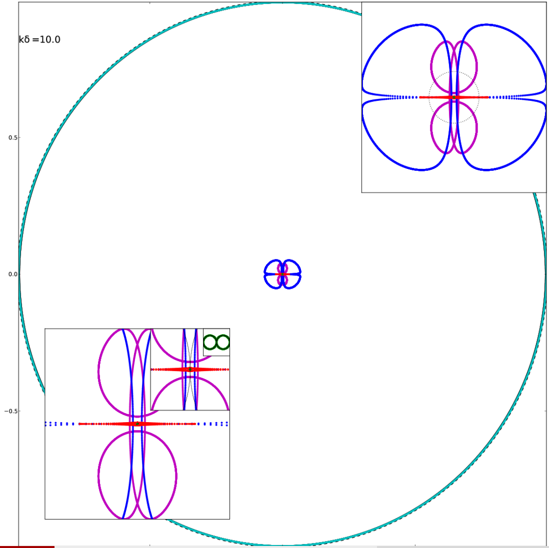


$\Rightarrow$ **animate through wavenumber range**

- similar obtain the ray (group) diagrams, again implicit derivation on 12th order DR, with limits short (EM) and long wavelength (MHD), group speeds $< c$! **animate through wavenumber**



- suppose you resolve up to $k\delta = 10$, interference leads to:



Take Home

- MHD to Hall-MHD to 2-fluid model: increasing complexity in wave dynamics: **dispersion rules**, enormous differences in wave propagation characteristics

  $\Rightarrow$ regime $k\delta \sim \mathcal{O}(0.1 - 10)$: fascinating constructive-destructive interferences
- can study all limits of physical relevance:
  - $\Rightarrow$ cold plasmas, electron-positron mixtures, ...
  - $\Rightarrow$ limit to Hall-MHD from 2-fluid: take $\mu \to 0$, $c \to \infty$
  - $\Rightarrow$ MHD as non-dispersive, long wavelength limit