No unique solution to the seismological problem of standing kink MHD waves.

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FWO Vlaanderen

1.Damped standing and propagating transverse MHD waves

- **Standing transverse MHD waves in coronal loops (Schrijver et al. (1999); Aschwanden et al. (1999)).**
- **Propagating transverse MHD waves (COMP (Tomczyk et al. 2007), SDO/AIA (McIntosh et al. 2011)).**
- **Transverse waves = kink waves $(m = 1)$ (Nakariakov et al. (1999).**
- **Rapid damping: resonant damping ? definitely at work.**
- **Exponential damping (eigenmode analysis) / Gaussian damping.**

For that's the attraction of the conference circuit: it's a way of converting work into play, combining professionalism with tourism. Write a paper and see the world.
Small World
David Lodge

2. Resonantly damped MHD waves on flux tubes / Eigenmodes.

2.1 Equilibrium and spatial and temporal dependence of waves

- **Cylindrical plasma column in static equilibrium.**
- **Loop of radius R:** $0 \leq r \leq R : i; r \geq R : e$
- **System of cylindrical coordinates** (r, z, φ)**.**
- **Equilibrium magnetic field** $\vec{B}_0 = B_0 \, \vec{1}_z$**, equilibrium density** $\rho_0(r)$**.**
- **Wave variables.**
- $\exp(-i\omega t), \quad \exp(i(m\varphi + k_z z)) \qquad m, k_z =$ **azimuthal and axial wave numbers.**
- **Kink modes** $m = 1$
- **Eigenmodes: dispersion relation DR**$(\omega, m, k_z, n_R) = 0.$

Eigenmodes: exponential damping

2.2 Resonant damping of kink MHD waves due to non-uniformity.

- **Inhomogeneity in interval** $b = R - l/2 \leq r \leq a = R + l/2$
- ρ_0 **varies from** ρ_i **to** ρ_e **in interval** $[R - l/2, \;\; R + l/2]$**.**
- $\omega_A(r) =$ **local Alfvén frequency**

$$\omega_A^2(r) = \frac{(\vec{k} \cdot \vec{B})^2}{\mu \rho_0(r)} \; = \; k_z^2 \, v_A^2(r)$$

- $v_A =$ **the local Alfvén velocity,** $v_A^2(r) = B^2/(\mu \, \rho_0(r))$.
- ω_A **varies from** ω_{Ai} **to** ω_{Ae} **in interval** $[R - l/2, \;\; R + l/2]$**.**
- **See e.g. Fig 4 of ER1983**

$$v_{Ai} < \omega/k_z < v_{Ae}, \;\; \omega_{Ai} < \omega < \omega_{Ae}$$

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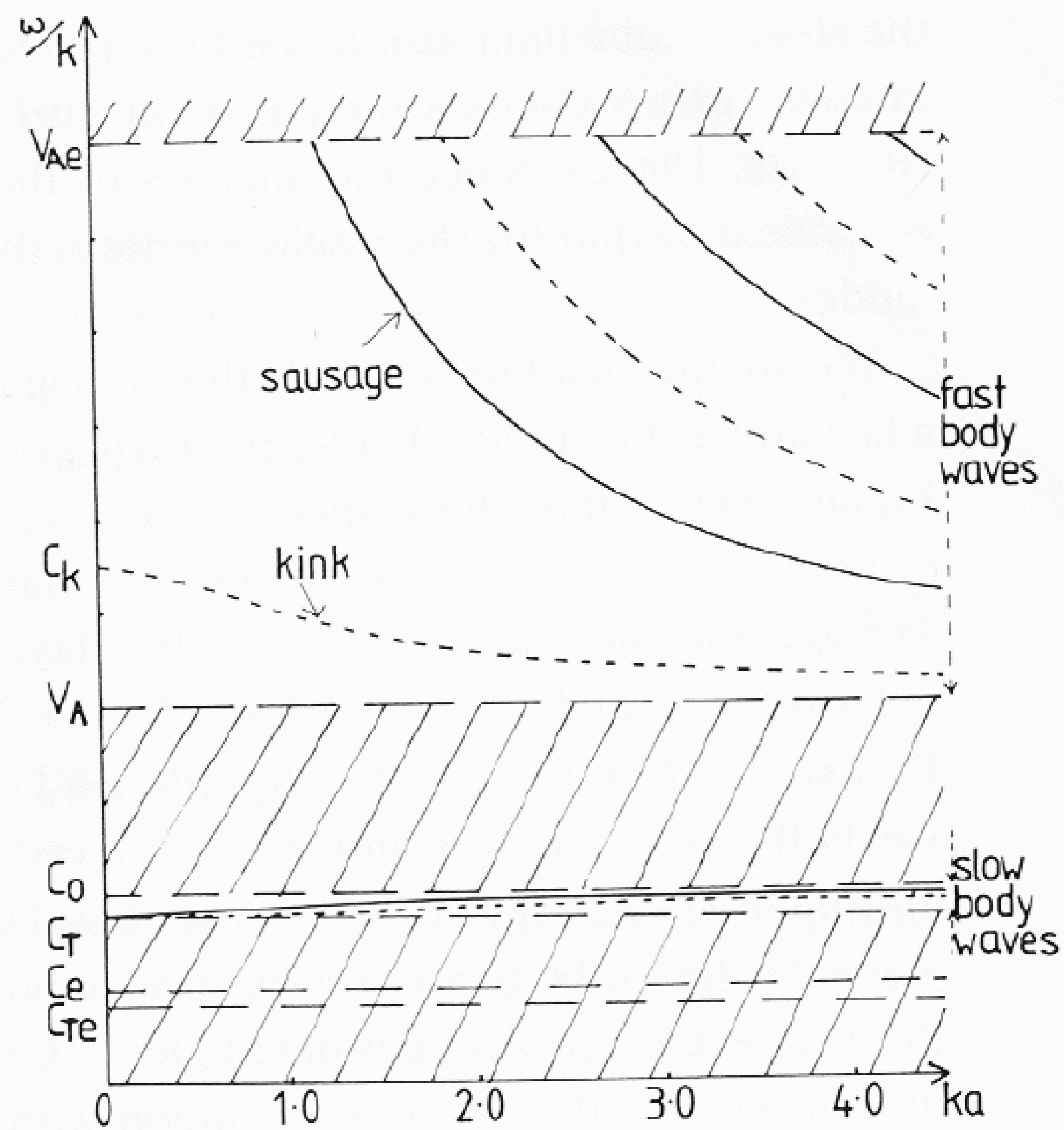
- **Replace the discontinuous variation of** ω_A **by a continuous variation.**
- **Classic kink** $(m = 1)$ **mode is always resonantly damped.**



ω/k
V_Ae
sausage
fast
body
waves
C_k
kink
V_A
C_0
C_T
C_e
C_Te
slow
body
waves
0
1·0
2·0
3·0
4·0
ka

- Non-axisymmetric ($m \geq 1$) waves with ω in AC undergo resonant damping.
- Standing waves: real wave number k_z: complex frequencies ω.

$$\exp(-i\omega t), \quad \omega = \omega_R + i\gamma$$
$$\exp(-i\omega t) = \exp(-i\omega_R t)\exp(\gamma t) = \exp(-i\omega_R t)\exp(-t/\tau_d)$$

- γ = the decrement, $\tau_d = 1/\mid\gamma\mid =$ the damping time.
- Phase velocity $v_{ph} = \omega/k_z$.

Eigenmodes: exponential damping

3. Direct problem for exponential damping.

3.1 How to compute resonantly damped eigenmodes.

- P and τ_D are determined by $\rho_0(r), p_0(r)$ and B_z .
- Direct problem: $\rho_0(r), p_0(r)$ and B_z are specified a priori.
- P and τ_D are computed.
- Numerical code / Generalized Frobenius method / Connection formulae.

3.2 Analytic dispersion relation

- **Use connection formulae: SGH-method.**
- **The thin boundary (TB) approximation: $l/R << 1$.**
- **The thin tube (TT) approximation.**
  - **Standing waves (long wavelength approximation): $k_z R << 1$.**
- **Dispersion Relation / Eq 74 of GHS1992.**

$$\rho_e(\omega^2 - \omega_{Ae}^2) + \rho_i(\omega^2 - \omega_{Ai}^2) = i\pi \frac{m/R}{\rho(r_A) \mid \Delta \mid} \ \rho_i(\omega^2 - \omega_{Ai}^2)\rho_e(\omega^2 - \omega_{Ae}^2)$$

-

$$\Delta = \frac{d}{dr}(\omega^2 - \omega_A^2(r)) \mid_{r_A}$$

- Δ $=$ **key quantity.**
- **Dispersion Relation relates frequency ω and wave number k_z.**

Standing waves.

- **Fix k_z and solve for ω (GHS1992, RR2002, GTAAB2009).**

$$\omega_R^2 = \frac{\rho_i\,\omega_{A,i}^2 + \rho_e\,\omega_{A,e}^2}{\rho_i + \rho_e} = \omega_k^2 = k_z^2\,v_k^2$$

$$\gamma = -\frac{\mid m \mid \pi}{2R}\,\frac{\rho_i^2\rho_e^2}{\rho(r_A)(\rho_i+\rho_e)^3}\,\frac{(\omega_{A,e}^2 - \omega_{A,i}^2)^2}{\mid \omega_k \mid\mid \Delta \mid} = \textbf{Eq. } 77 \textbf{ GHS}1992$$

- v_k **is kink velocity** $v_k^2 = (B_i^2 + B_e^2)/(\mu(\rho_i + \rho_e))$

Period

- **Equal and constant magnetic fields** $B_i = B_e = B$

$$P = \tau_{Ai}\,\sqrt{2}\,\left\{\frac{\zeta+1}{\zeta}\right\}^{1/2}$$

- $$\zeta = \rho_i/\rho_e,\quad \tau_{Ai} = L/v_{Ai},\quad v_{Ai} = B/\sqrt{\mu\rho_i},\quad v_k = \sqrt{2}\,v_{Ai}\,\left(\frac{\zeta}{\zeta+1}\right)^{1/2}$$

- $L =$ **the length of the loop.**

Exponential damping rate γ and exponential time τ_D

- **TTTB approximation: Effect of the non-uniform layer on the damping is contained in the value of Δ. i.e. the derivative of $\omega_A^2(r)$ at r_A .**
- **Take equal and constant magnetic fields $B_i = B_e = B$ in Eq. 77 GHS1992.**
- **See equation (31) of Goossens et al. 2009**

$$\frac{\gamma}{\omega_k} = -\frac{\pi}{8} \frac{\mid m \mid}{R} \frac{(\rho_i - \rho_e)^2}{(\rho_i + \rho_e)} \frac{1}{\mid (\frac{d\rho}{dr})_{r_A} \mid}$$

- **Effect of the non-uniform layer is determined by**

$$(\frac{d\rho_0}{dr})_{r_A}$$

- **Characterize the variation of density in the non-uniform layer at the pos-tion r_A by quantity G**

$$G = \frac{R}{(\rho_i - \rho_e)} \mid (\frac{d\rho_0}{dr})_{r_A} \mid$$

- **Damping rate and damping time τ_D**

$$\frac{\gamma}{\omega_k} = -\frac{\pi}{8}\frac{(\rho_i-\rho_e)}{(\rho_i+\rho_e)}\frac{1}{G}, \quad \frac{\tau_D}{P} = \frac{4}{\pi^2}\frac{\zeta+1}{\zeta-1}G$$

- **For a given ζ, τ_D/P is determined by the dimensionless quantity G.**
- **Direct problem: the equilibrium configuration is freely choosen.**
- **Confine variation of density to a layer of thickness l with steepness α (VD et al. 2004)**

$$\frac{d\rho_0}{dr}\,|_R = -\alpha\frac{\rho_i-\rho_e}{l}$$

- **Linear variation: $\alpha = 1$; sinusoidal variation (RR 2002) $\alpha = \pi/2$.**
-

$$G = \frac{\alpha}{(l/R)}$$

- **Same value of τ_D/P for infinitely many couples $(\alpha, l/R)$.**
- **$\{(\tau_D/P)_L$ for a layer with thickness $(l/R)_L$ $\}$ = $\{(\tau_D/P)_S$ for a layer with thickness $(l/R)_L \times \pi/2$ $\}$ and vice versa.**

- **Direct problem:** α **and** l/R **are prescribed.**
- **Inverse problem: cannot distinguish between different couples of** $(\alpha, l/R)$.

$$\frac{\tau_D}{P} = \frac{1}{\mid m \mid} \frac{4}{\pi^2} \frac{\alpha}{l/R} \frac{\zeta+1}{\zeta-1}$$

- $\alpha = 1$ **for linear variation (see Eq. 79b GHS 1992),** $\alpha = \pi/2$ **for a sinusoidal variation (see RR 2002).**
- **Big success! Explains rapid damping of standing transverse waves!**
- **Example** $\alpha = \pi/2, \quad \rho_i/\rho_e = 5, \quad l/R = 1/4, \quad \tau_D/T = 12/\pi \approx 4$
- **Fast damping predicted about a decade before it was observed.**

4. Direct problem for Gaussian damping.

4.1 Propagating waves.

- **First studied by Pascoe et al. 2012 in numerical simulations.**
- **Hood et al. 2013: Analytical theory for propagating waves.**
- **TTTB approximation and a linear variation of density: $\alpha = 1$.**
- **Analytical expression for the Gaussian damping length L_G.**

$$L_G^2 = \frac{16}{l/R} \frac{1}{k_z^2} \left(\frac{\zeta + 1}{\zeta - 1} \right)^2$$

- **Pascoe et al. 2013: expression for height h_S of the switch of the Gaussian to the exponential profile.**

4.2 Standing waves.

- **Analytical theory for temporal Gaussian damping of standing waves by Ruderman and Terradas 2013.**
- **TTTB approximation and a linear variation of density ($\alpha = 1$).**
- **No analytical expressions for τ_G, t_S.**
- **MVD2016: Numeric simulations / Pascoe et al. 2016: Observations**

5. Seismology for standing waves of coronal loops.

- **Initial evolution: Gaussian damping.**
- **No theoretical expressions on Gaussian damping of standing waves**

"I've done everything the Bible says. Even the stuff that contradicts the other stuff. What more can I do? "
N.F. **The Simpsons**

5.1 Limitations.

- **Period P and the damping times τ_D, τ_G are determined by: $\rho_0(r), p_0(r), \vec{B}_0(r)$.**
- **Helio-seismology is a big success.**
- **Big difference between helioseismology and seismology of coronal loops.**
- **Seismology of coronal loops is not able to determine the radial structure of equilibrium.**
- **Only a few characteristic quantities can be determined. No unique solution for equilibrium.**
- **Reduced seismological problem.**

5.2 Many solutions

- Recall equations for P and τ_D

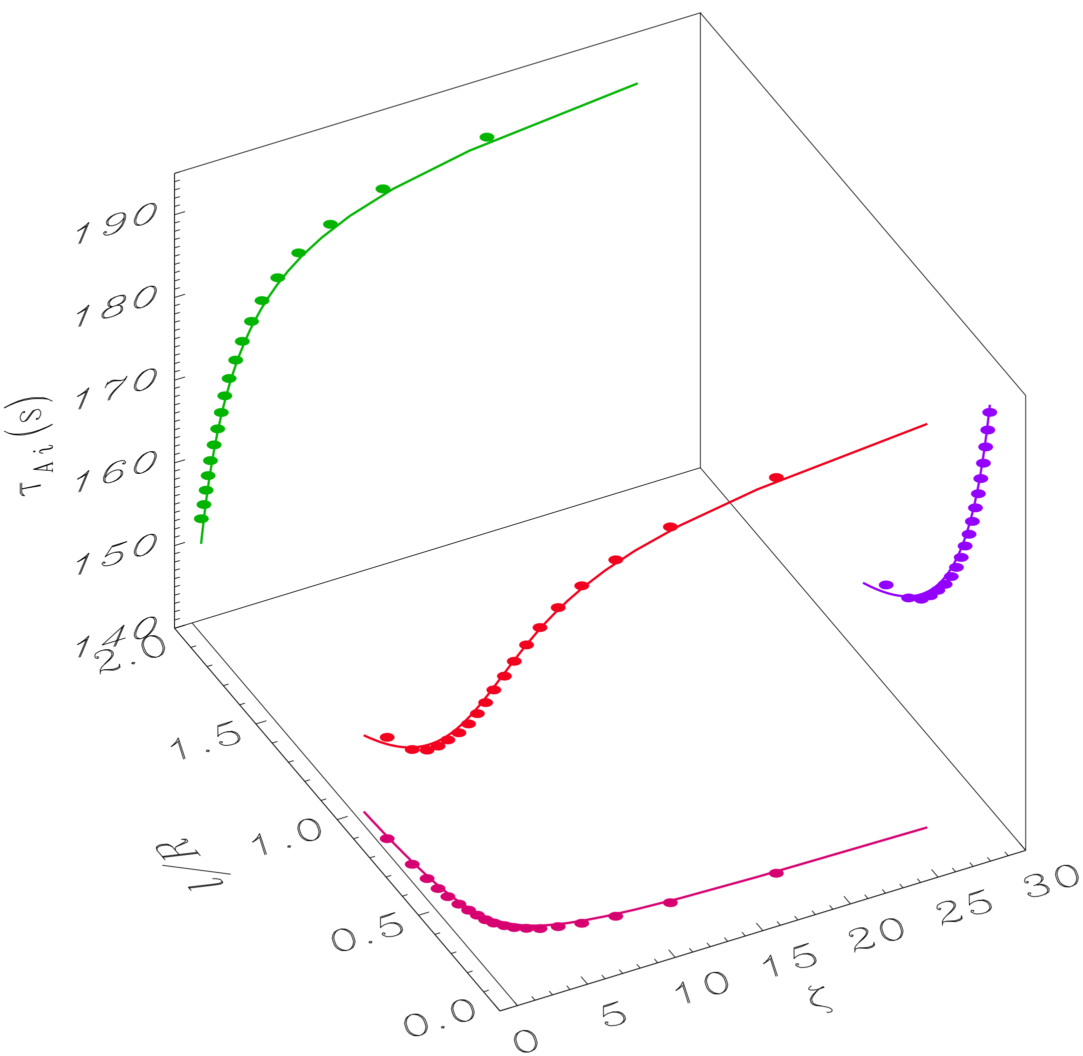
$$\frac{P}{\tau_{Ai}} = \sqrt{2}\left\{\frac{\zeta+1}{\zeta}\right\}^{1/2}, \quad \frac{\tau_D}{P} = \frac{4}{\pi^2}\frac{\zeta+1}{\zeta-1}G, \quad G = \frac{\alpha}{(l/R)}$$

- 2 observables: period P and damping time τ_D
- 4 unknown equilibrium quantities: $\zeta, \tau_{Ai}, l/R, \alpha$
- 2 equations for 4 unknowns: an underdetermined system.
- ∞^2 solutions

5.3 Different cases

- Observed values of period.
  - 1 equation and 2 unknowns: $\zeta = \rho_i/\rho_e$ and $\tau_{Ai} = L/v_{Ai}$.
  - ∞^1 solutions.
  - Nakariakov and Ofman 2001: $\rho_i/\rho_e = 10$; single solution for τ_{Ai}.
  - Choose value for L and ρ_i and determine an estimate for B.

- **Observed values of damping times.**
  - **1 equation and 3 unknowns:** ζ**,** l/R **and** α **(**$G = \alpha/(l/R)$ **).**
  - ∞^2 **solutions.**
  - **Prescribe** α **and** ζ **and find single solution for** l/R **or vice versa.**
  - **Ruderman and Roberts, 2002 and Goossens et al. 2002 :** $\rho_i/\rho_e = 10$ **and** $\alpha = \pi/2$ **(sinusoidal variation).**
- **Observed values of periods and damping times.**
  - **2 equations and 4 unknowns:** $\tau_{Ai} = L/v_{Ai}$**,** ζ**,** l/R**, and** α**.**
  - ∞^2 **solutions.**
  - **Arregui et al. 2007 and Goossens et al. 2008 :** $\alpha = \pi/2$**.**
  - **2 equations for 3 unknowns:** ∞^1 **solutions.**
  - **Recover results by the use of a linear variation of density and non-uniform layers with thickness** $(l/R)_L = (l/R)_S \times (2/\pi)$**.**
- **Reduced seismology:** ∞ **solutions.**
- **Single solution : prescribe values of unknown quantities, e.g.** $\alpha,\ \rho_i/\rho_e$.



190
180
170
160
150
140
$\tau_{Ai}(s)$
2.0
1.5
1.0
0.5
0.0
l/R
0
5
10
15
20
25
30
ζ

6. Bayesian Inference for exponential time damping

6.1 Inference for ζ and l/R

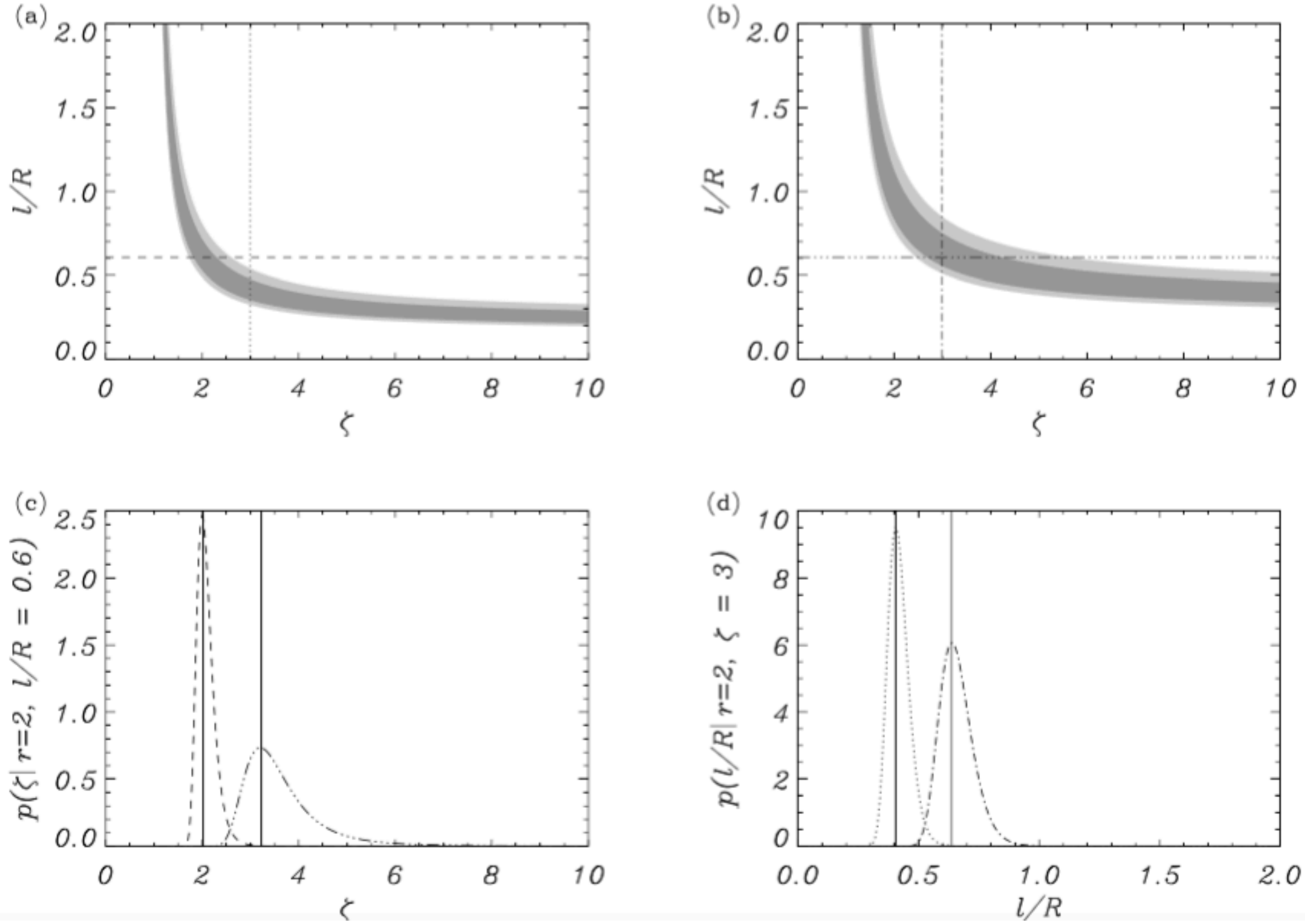
$$\frac{\tau_D}{P} = \frac{4}{\pi^2}\,\frac{\alpha}{l/R}\,\frac{\zeta+1}{\zeta-1} = \frac{4}{\pi^2}\,G\,\frac{\zeta+1}{\zeta-1}$$

- **2-dimensional joint probability density function.**
- **1-dimensional cuts for fixed l/R and fixed ζ.**
- **Marginal posteriors for ζ and l/R**

6.2 Inference for α and G

$$\frac{\tau_D}{P} = \frac{4}{\pi^2}\frac{\zeta+1}{\zeta-1}\,G, \quad G = \frac{\alpha}{(l/R)}$$

- **Marginal posteriors for α and G**



(a)
l/R
ζ
0.0
0.5
1.0
1.5
2.0
0
2
4
6
8
10
(b)
l/R
ζ
0.0
0.5
1.0
1.5
2.0
0
2
4
6
8
10
(c)
p(ζ| r=2, l/R = 0.6)
ζ
0.0
0.5
1.0
1.5
2.0
2.5
0
2
4
6
8
10
(d)
p(l/R| r=2, ζ = 3)
l/R
0
2
4
6
8
10
0.0
0.5
1.0
1.5
2.0

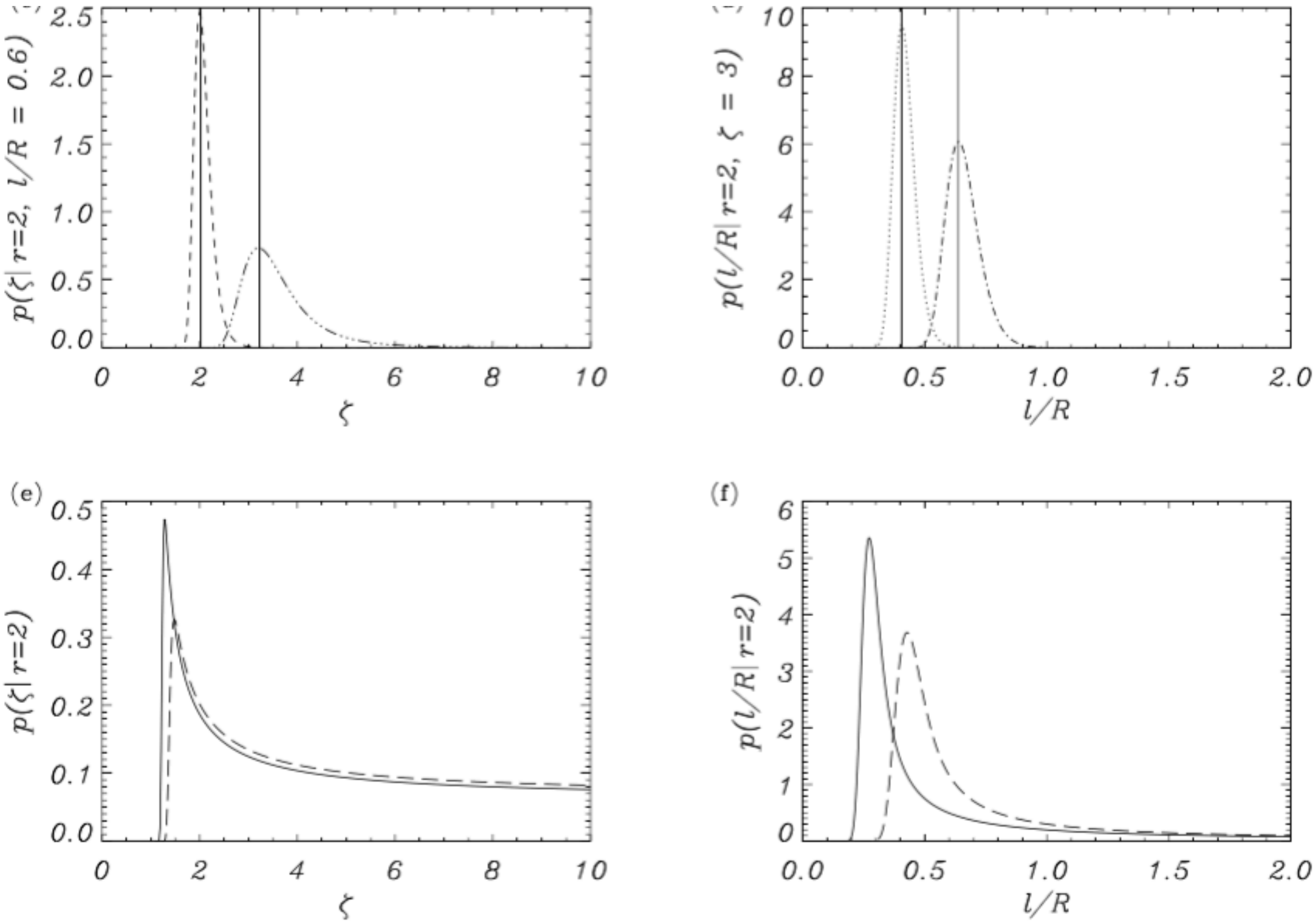


Fig. 1. Results from the comparison of inferences using two values of α in Eq. (16). (a) and (b) show the two-dimensional joint posteriors in the $(\zeta, l/R)$ parameter space for $\alpha = 1$ and $\alpha = \pi/2$, respectively. (c) and (d) show one-dimensional cuts of the joint posteriors along fixed values of $l/R = 0.6$ and $\zeta = 3$, respectively. Same line-styles are used to identify cut directions in (a) and (b) with the corresponding results in (c) and (d). The vertical solid lines in (c) and (d) show the algebraic inversion results using Eq. (16) and the fixed values of $l/R = 0.6$ in panel (c) and $\zeta = 3$ in panel (d). (e) and (f) show the marginal posteriors for ζ and l/R, respectively. In these calculations $r = \tau_D/P = 2$ and $\sigma = 0.1r$.

6.2 Inference for α and G.

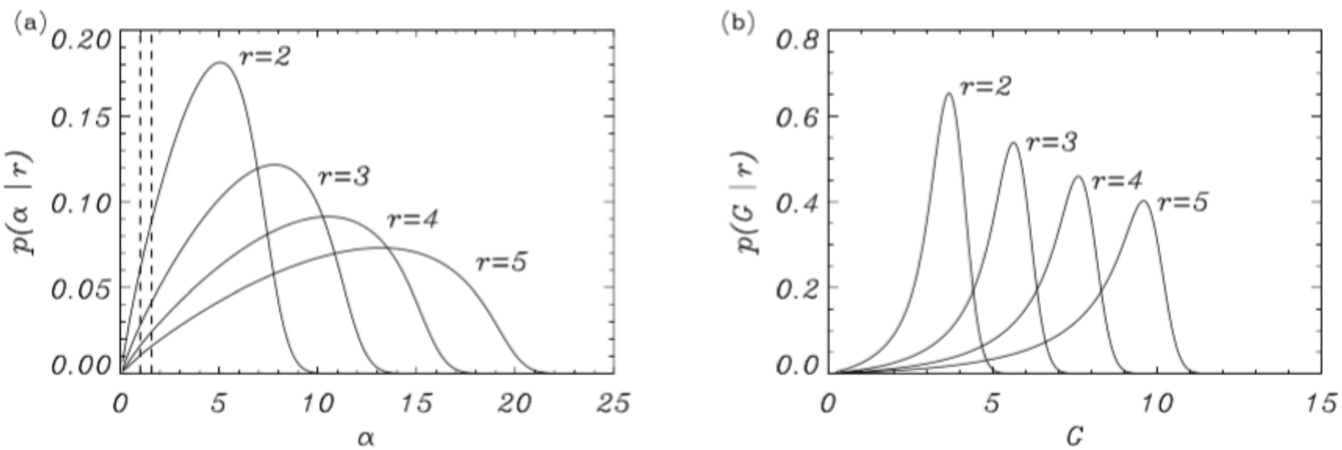


Fig. 3. Marginal posteriors for (a) α and (b) G for different values of the damping ratio $r = \tau_d/P$ obtained by inversion of Eqs. 16 and (13) and by considering them as additional parameters together with ζ and l/R. In (a) the vertical dashed lines indicate the values $\alpha = 1$ (linear density model) and $\alpha = \pi/2$ (sinusoidal density model). The mean and error at 68% credible intervals for the posteriors in (a) are: $\alpha = 4.6^{+1.9}_{-2.3}$ for $r = 2$; $\alpha = 6.9^{+2.8}_{-3.3}$ for $r = 3$; $\alpha = 9.2^{+3.8}_{-4.4}$ for $r = 4$; and $\alpha = 11.5^{+4.8}_{-5.5}$ for $r = 5$. The mean and error at 68% credible intervals for the posteriors in (b) are: $G = 3.5^{+0.6}_{-0.8}$ for $r = 2$; $G = 5.3^{+0.7}_{-1.2}$ for $r = 3$; $G = 7.1^{+0.8}_{-1.7}$ for $r = 4$; and $G = 8.9^{+0.9}_{-2.1}$ for $r = 5$. In all computations the error in damping ratio is fixed to $\sigma = 0.2$.

7. Conclusions

- **Reduced seismology.**
- **Characteristic quantities.**
- ∞ **many solutions even to the reduced problem.**
- **Single solution: prescribe values for unknown quantities.**
- **Bayesian analysis.**
- **Gaussian damping of standing waves: analytical expressions for** $\tau_G, \ t_S$

"Has anything escaped me?" I asked with some self-importance.
"I trust that there is nothing of consequence which I have overlooked?"
"I am afraid my dear Watson, that most of your conclusions were erroneous."
The Hound of the Baskervilles.
A. Conan Doyle.